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ON FACTORIZATION OVER $k((z))[\delta]$

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Abstract. The article is conceived to have a background in order to obtain an algorithmic method for the formal solutions of a linear differential equation. The solving method is based on a factorization of the differential operators, proposed by using the Newton polygon of a linear differential operator. A subclass of this class of equations is completely solved in the end of the paper.

1. Introduction

Suppose we want to solve a linear differential equation with coefficients in $k(z)$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{1}$$

We assign to (1) the differential operator:

$$D = a_n \frac{d^n}{dz^n} + a_{n-1} \frac{d^{n-1}}{dz^{n-1}} + \cdots + a_1 \frac{d}{dz} + a_0. \tag{2}$$

After factorizing (2) will be much easier to solve (1). Because the polynomial in $\frac{d}{dz}$ described by (2) is not in a commutative ring, the usual Hensel lifting cannot be performed.

Example. Let be the second order linear differential equation

$$y'' - \frac{1}{z+1/2} y' = 0, \tag{3}$$

with 1, $z + z^2$ as solutions (they are not unique).

One can assign the operator

$$D = \frac{d^2}{dz^2} - \frac{1}{z+1/2} \frac{d}{dz}.$$

Try to factorize it!

Because 1 is a solution of (3) we may think to a factorization of the type $D = D_1 \frac{d}{dz}$. Here $D_1 = \frac{d}{dz} - \frac{1}{z+1/2}$. Thus we have one factorization

$$D = (\frac{d}{dz} - \frac{1}{z+1/2})(\frac{d}{dz}).$$

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Verify that indeed $z + z^2$ is also a solution of the factored equation.

An important remark to make here is that $D_1 = \frac{d}{dz} - \frac{1}{z+1/2}$ and $D_2 = \frac{d}{dz}$ will not commute. This means that if we rewrite $D = D_2 D_1$, the solutions 1 and $z + z^2$ will not verify $D \equiv 0$ anymore. Moreover $D = D_1 D_2$ is not the unique factorization. For example

$$D = (\frac{d}{dz} + \frac{2z+1}{z^2+z} - \frac{2}{2z+1})(\frac{d}{dz} - \frac{2z+1}{z^2+z})$$

gives another decomposition of D. Check the solutions!

2. The Newton polygon of a linear differential operator

We will try to give a factorization of a special class of differential operators, using the Newton polygon and its properties.

Notations. Let be k an algebraically closed field, with $char(k) = 0$ and define $\mathrm{D} = k((z))[\delta]$ a linear differential operator with Laurent series in z as coefficients. Here $\delta = z\frac{\partial}{\partial z}$ is defined to preserve the powers of z:

$$L = \sum_{i=0}^{n} a_i \delta^i, \quad a_i \in k[[z]], \; a_n \neq 0, \quad a_i = \sum_{j=-\infty}^{\infty} a_{ij} z^j.$$

If $a_n = 1$ we say that the differential operator in δ is *monic*.

Remark 2.1. $\delta z = z\delta + z$ *(see the noncommutativity).*

Definition 2.2. *For* $m \in \mathbf{Z}$, $n \in \mathbf{N}$ *call* $z^m \delta^n$ monomial.

Definition 2.3. *The* order *on the monomials:*

$$z^{m_1}\delta^{n_1} \geq z^{m_2}\delta^{n_2} \Leftrightarrow (m_1 \geq m_2, n_1 \leq n_2).$$

Note that this is a partial ordering.

Remark 2.4. *Define* $(n_1, m_1) \geq (n_2, m_2) \Leftrightarrow (n_1 \leq n_2, m_1 \geq m_2)$. *Thus* $z^{m_1}\delta^{n_1} \geq z^{m_2}\delta^{n_2} \Leftrightarrow (n_1, m_1) \geq (n_2, m_2)$.

Definition 2.5. *The* Newton polygon $N(L)$ *of a linear differential operator* $L \in \mathrm{D}$ *is the convex hull of the set*

$$W = \{(x, y) \in \mathbf{R}^2 | \exists z^m \delta^n \in L, (x, y) \geq (n, m)\}.$$

Denote by $(m_1, n_1), (m_2, n_2), \ldots, (m_r, n_r)$ *the vertices of the Newton polygon, including here the special point* $(m, 0)$. *A slope of a Newton polygon is given by* $k_i = \frac{m_{i+1} - m_i}{n_{i+1} - n_i}$ *and the length of the slope* k_i *is* $n_{i+1} - n_i$.

Now, we can define a partial ordering for differential operators of $\mathbf{D}$ and we say that $L_1 \geq L_2$, $L_1, L_2 \in \mathbf{D}$ if all the terms of L_1 are inside of the Newton polygon of L_2.

Example 2.6. *Mark van Hoeij's example.*

$$\begin{array}{rcllll} L & = & 7z^{-5} & +2z^{-6}\delta & +2z^{-5}\delta & +3z^{-5}\delta^2 \\ & & -3z^{-5}\delta^3 & +5z^{-4}\delta^3 & +z^{-4}\delta^5 & +2z^{-2}\delta^5 \\ & & +2z^{-3}\delta^6 & +3z^{-2}\delta^7 & +2z^{-1}\delta^8 & +\delta^9. \end{array}$$

We see that there are no negative slopes allowed, but if our ring were commutative we got the negative slope.

Definition 2.7. *The vertices in the Newton polygon are called* extremal points.

The idea of factoring a linear differential operator is contained in the slopes of the Newton polygon; permuting slopes gives other factorizations and moreover, the factors are different for each permutation of slopes.

Definition 2.8. *Let $b(L)$ be the graph of $N(L)$. The* boundary part *of L is:*

$$B(L) = \sum_{(n,m)\in b(L)} a_{n,m} z^m \delta^n.$$

Example 2.9. *In the example above:*

$$B(L) = 2z^{-6}\delta + z^{-4}\delta^5 + 2z^{-3}\delta^6 + 3z^{-2}\delta^7 + 2z^{-1}\delta^8 + \delta^9.$$

Notation. $R(L) := L - B(L)$ and it is called the *interior part* of L.

Remark 2.10. *$R(L) > B(L)$ and $R(L) > L$.*

Proposition 2.11. *If $M_1 = z^{m_1}\delta^{n_1}$, $M_2 = z^{m_2}\delta^{n_2}$ then*

$$M_1 M_2 = z^{m_1+m_2}(\delta + m_2)^{n_1}\delta^{n_2} \tag{4}$$

$$B(M_1 M_2) = z^{m_1+m_2}\delta^{n_1+n_2}. \tag{5}$$

Proof. First part can be done by induction and it is left as an exercise. For the second part we expand the term $(\delta + m_2)^{n_1}$ and since the powers of δ are decreasing from $\delta^{n_1+n_2}$ it is obvious that the boundary part of $M_1 M_2$ will be $z^{m_1+m_2}\delta^{n_1+n_2}$. □

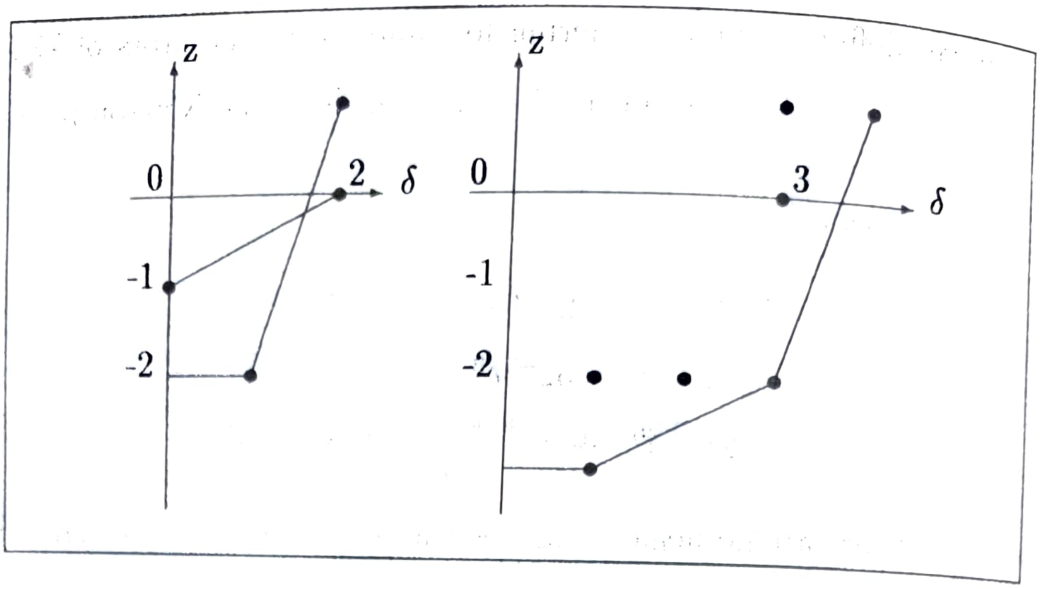

FIGURE 1. Example

Example 2.12. *Let be two operators $L_1 = z^{-1} + \delta^2$ and $L_2 = z^{-2}\delta + z\delta^2$. Then*

$$L = L_1L_2 = z^{-3}\delta + \delta^2 + z^{-2}\delta^3 - 4z^{-2}\delta^2 + 4z^{-2}\delta + +2z\delta^3 + z\delta^4.$$

The interesting thing here are the slopes of L: they include all the slopes of L_1 and L_2, the length of slopes in L_1L_2 is the sum of lengths of the same slopes in L_1 and L_2 and moreover, the Newton polygon of L is the sum of Newton polygons of L_1 and L_2. We will prove this in the next lemma.

Lemma 2.13. *Let be $L_1, L_2 \in \mathbf{D}$. Then*

$$N(L_1L_2) = N(L_1) + N(L_2). \tag{6}$$

Proof. Take

$$L_1 = \sum_{i,j} a_{i,j}z^j\delta^i, \quad L_2 = \sum_{i,j} b_{i,j}z^j\delta^i.$$

Then

$$L_1L_2 = \sum_{\substack{(i_1,j_1)\\(i_2,j_2)}} a_{i_1,j_1}b_{i_2,j_2}z^{j_1+j_2}(\delta + j_2)^{i_1}\delta^{i_2}.$$

The terms $z^{j_1+j_2}(\delta + j_2)^{i_1}\delta^{i_2}$ are contained in the Newton polygon $N(L_1) + N(L_2)$. Thus $N(L_1L_2) \subseteq N(L_1) + N(L_2)$.

Now show that extremal points of $N(L_1)+N(L_2)$ are in $N(L_1L_2)$. So, if (s_2,s_2) is extremal in $N(L_1)+N(L_2)$ then there exist unique $(i_1,j_1)\in b(L_1)$ and $(i_2,j_2)\in b(L_2)$ such that

$$(s_1,s_2)=(i_1,j_1)+(i_2,j_2).$$

Hence $z^{j_1+j_2}\delta^{i_1+i_2}\in N(L_1L_2)$. From this it follows the other inclusion, $N(L_1)+N(L_2)\subseteq N(L_1L_2)$. □

Remark 2.14. *The boundary part of* $L=\sum_{\substack{(i_1,j_1)\in b(L_1)\\(i_2,j_2)\in b(L_2)}} a_{i_1,j_1}b_{i_2,j_2}z^{j_1+j_2}\delta^{i_1+i_2}$ *is*

$$B(L)=\sum_{(s_1,s_2)\in b(L_1L_2)}\left(\sum_{(i_1,j_1)+(i_2,j_2)=(s_1,s_2)} a_{i_1,j_1}b_{i_2,j_2}\right)z^{s_2}\delta^{s_1}. \tag{7}$$

Take this example, now:

$$\begin{aligned} L_1 &= z^{-1}+\delta+terms\ \text{"}inside\text{"}\ N(L_1),\\ L_2 &= z^{-1}\delta+\delta^2+terms\ \text{"}inside\text{"}\ N(L_2). \end{aligned}$$

The extremal points for L_1 are $(0,-1),(1,0)$; for L_2 are $(1,-1),(2,0)$.

$$L_1L_2=(z^{-2}-z^{-1})\delta+2z^{-1}\delta^2+\delta^3+terms\ \text{"}inside\text{"}\ N(L_1L_2).$$

Here, the extremal points are $(1,-2),(3,0)$. But the all combinations of monomials used in (7) are not exhausted. For example we also have

$$\begin{aligned} (0,-1)+(2,0) &= (2,-1)\\ (1,0)+(1,-1) &= (2,-1)\ (see\ point\ P). \end{aligned}$$

So, if (s_1,s_2) is given in a not unique way, then exists common slopes of L_1 and L_2. Moreover, (s_1,s_2) is not an extremal point.

Corollary 2.15. *If* $L_1,L_2\in\mathbf{D}$ *then:*

(i) The set of slopes of L_1L_2 *is the union of the slopes of* L_1 *and* L_2.

(ii) The length of slopes in L_1L_2 *is the sum of lengths of the same slopes in* L_1 *and* L_2.

Proof. (i) By (6) and geometry of $N(L_1)+N(L_2)$.

(ii) Using again (6) and the result of (i). □

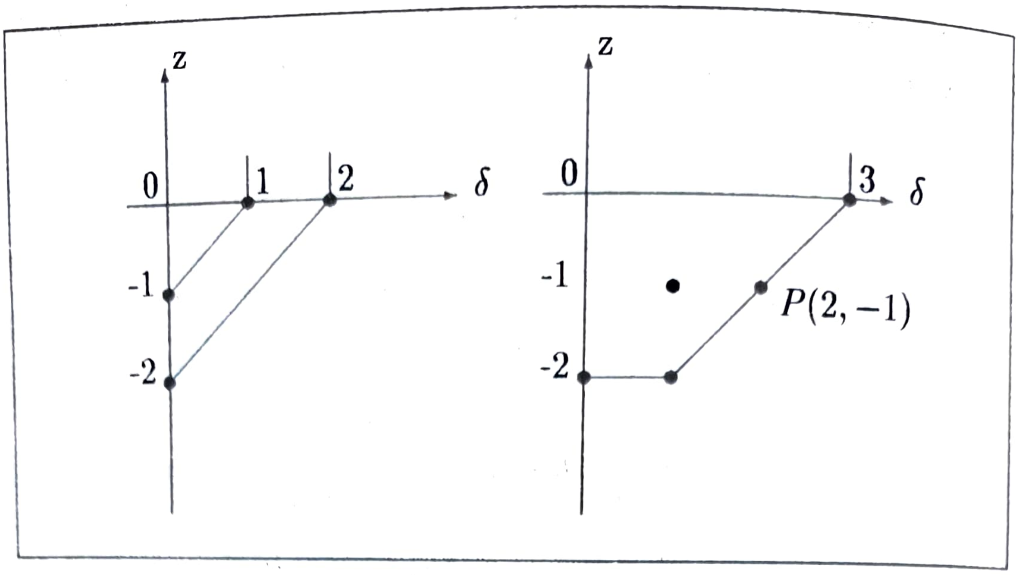

FIGURE 2. The example from the **Remark**

Example 2.16. *Recall the example from the begining of this paper:* $L = \frac{d^2}{dz^2} - \frac{1}{z+1/2}\frac{d}{dz}$, *the linear differential operator of equation* (3). *We have given there two different factorizations:*

$$L = L_1'L_2',\ L_1' = \frac{d}{dz} - \frac{1}{z+1/2},\ L_2' = \frac{d}{dz}$$
$$L = L_1''L_2'',\ L_1'' = \frac{d}{dz} + \frac{2z+1}{z^2+z} - \frac{2}{2z+1},\ L_2'' = \frac{d}{dz} - \frac{2z+1}{z^2+z}.$$

Rewriting everything in terms of δ we obtain

$$L_1' = z^{-1}\delta - \frac{1}{z+1/2} = \frac{1}{z+1/2}((1+\frac{1}{2}z^{-1})\delta - 1),$$
$$L_2' = z^{-1}\delta,$$
$$L_1'' = \frac{1}{z(2z+1)(z+1)}((2z^2+3z+1)\delta + 2z^2 + 2z + 1),$$
$$L_2'' = \frac{1}{z+z^2}((1+z)\delta - 2z - 1).$$

Let's state an important theorem, which can be proved by building up from the simpler cases.

Theorem 2.17. *Let be $L \in \mathbf{D}$ with its Newton polygon, $N(L)$ having the slopes $k_1, k_2, \ldots, k_r$. Then there is a factorization*

$$L = L_1 L_2 \cdots L_r,$$

where $N(L_i)$ has unique slope k_i, for each $i \in \{1, 2, \ldots, r\}$. (8)

Case of one slope

We will start first from $L = L_1 M$, $M \in \mathbf{D}$, where L_1 has only one slope. The simplest case here is that one when the slope is 0. One can write for an $L \in$ D, $L = \sum_{i=-\infty}^{\infty} z^i L(i)(\delta)$, where $L(i) \in k[\delta]$ are polynomials in δ.

Take $n_1 \neq 0$ to be the length of the slope 0 of $N(L_1)$ and we want $L_1 = \sum_{i \geq 0} z^i L_1(i)(\delta)$, $L_1(0)$ is monic of degree n_1 and $L_1(i)$ has degree less than n_1, to maintain the unique slope 0. Also $L_2 = \sum_{i \geq m_1} z^i L_2(i)(\delta)$, where m_1 is the greatest degree occuring in L_1. Successively one can get

$$
\begin{aligned}
z^{-j} L_1(i)(\delta) z^j &= z^{-j} \sum_k a_{i,k} \delta^k z^j = z^{-j} z^j \sum_k a_{i,k} (\delta + j)^k = \\
&= \sum_k a_{i,k} (\delta + j)^k = L_1(i)(\delta + j).
\end{aligned}
$$

Now we compute the product $L_1 L_2$

$$
\begin{aligned}
L_1 L_2 &= \Big(\sum_{i \geq 0} z^i L_1(i)(\delta)\Big)\Big(\sum_{j \geq m_1} z^j L_2(j)(\delta)\Big) = \\
&= \sum_{i \geq 0} z^i \sum_{j \geq m_1} z^j z^{-j} L_1(i)(\delta) z^j L_2(j)(\delta) = \\
&= \sum_{i \geq 0} z^i \sum_{j \geq m_1} z^j L_1(i)(\delta + j) L_2(j)(\delta) = \\
&= \sum_{k \geq m_1} z^k \sum_{\substack{i+j = k \\ i \geq 0 \\ j \geq m_1}} L_1(i)(\delta + j) L_2(j)(\delta).
\end{aligned}
$$

But $L = \sum_{k \geq m_1} z^k L(k)(\delta)$, so

$$
\sum_{k \geq m_1} z^k L(k)(\delta) = \sum_{k \geq m_1} z^k \sum_{\substack{i+j = k \\ i \geq 0 \\ j \geq m_1}} L_1(i)(\delta + j) L_2(j)(\delta)
$$

For $k = m_1$ we have $L_1(0)(\delta + m_1) L_2(m_1)(\delta) = L(m_1)(\delta)$. For $k = m_1 + 1$, $L_1(0)(\delta + m_1 + 1) L_2(m_2 + 1)(\delta) + L_1(1)(\delta + m_1) L_2(m_1)(\delta) = L(m_1 + 1)(\delta)$. The last identity is nothing else but the division of $L(m_1 + 1)(\delta)$ by $L_1(0)(\delta + m_1 + 1)$, with the remainder $L_1(1)(\delta + m_1) L_2(m_1)(\delta)$ and the quotient to be $L_2(m_2 + 1)(\delta)$ and by the uniqueness of the division theorem $L_1(\delta + m_1)$ and $L_2(m_1+1)$ are uniquely determined. Doing by this way one can find the factorization of L, by lifting from $L(0) = L_1(0)L_2(m_1)$.

Let's start to say something about the case when the one slope of $N(L_1)$ is > 0. The ideea of this case is that it reduces to the previous case. Take this one slope to be the minimal slope of L and that is $m = b/a$, $(a,b) = 1$, $a, b \in \mathbf{Z}$. Replace δ by $\Delta = t^b\delta$, $z = t^a$. $\Delta = t^b\delta = \frac{t^{b+1}}{a}\frac{d}{dt}$. Thus $\Delta t = \frac{t^{b+1}}{a} + \frac{t^{b+2}}{a}\frac{d}{dt} = \frac{t^{b+1}}{a} + t\Delta$. The new Newton polygon will have one slope 0.

Example 2.18.

$$L = z\delta^2 - 1. \tag{9}$$

Take $m = 1/2, t = z^{1/2}, \Delta = t\delta$. So $\Delta t = \frac{t^2}{2} + t\Delta$ and $\Delta^2 = (t\delta)(t\delta) = t^2\delta^2 + \frac{1}{2}t\Delta$. Thus $t^2\delta^2 = \Delta^2 - \frac{1}{2}t\Delta$. Replacing $t^2\delta^2$ in (8) will give

$$L = \Delta^2 - \frac{1}{2}t\Delta - 1.$$

And thus the unique slope is zero.

References

[1] M. van Hoeij, *Formal solutions and factorization of differential operators with series coefficients*, Manuscript, Holland, 1995.

[2] A.H.M. Levelt, *Polynomials over finite fields*, Course at Uiversity of Nijmegen, Nijemgen, 1995.

[3] B. Malgrange, *Sur la reuction formelle des equations differentielles a singularites irregulieres*, Manuscript, France, 1979.

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