A GOOD DRAWING OF COMPLETE BIPARTITE GRAPH $K_{9,9}$, WHOSE CROSSING NUMBER HOLDS ZARANKIEWICZ CONJECTURES

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Abstract. There exist some Drawing for any graph $G = (V, E)$ on plan. An important aim in Graph Theory and Computer science is obtained a best drawing of an arbitrary graph. Also, a draw of a non-planar graph $G$ on plan generate several edge-cross. A good drawing (or strongly best drawing) of $G$ is consist of minimum edge-cross.

The crossing number of a graph $G$, is the minimum number of crossings in a drawing of $G$ in the plane, denoted by $cr(G)$. A crossing is a point of intersection between two edges. The crossing number of the complete bipartite graph is one of the oldest crossing number open problems.

In this paper, we present a good drawing of complete bipartite graph $K_{9,9}$. This drawing is able to developed on $K_{n,n}$, $8 \leq n \leq 9$ and implies that the crossing number of these graphs hold Zarankiewicz conjecture. $\forall n, m \in \mathbb{N}$ Zarankiewicz conjecture is equal to

$$cr(K_{n,m}) \leq Z(m, n) = \left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right].$$

1. INTRODUCTION

Let $G = (V, E)$ be a simple finite connected graph with the vertex set $V(G)$ and the edge set $E(G)$. $|V(G)| = n$, $|E(G)| = e$ are the number of vertices and edges.

For each vertex $v$ of a graph $G$, let $N_G(v) := \{u \in V(G)|uv \in E(G)\}$ be the neighborhood of $v$ in $G$. The degree of $v$, denoted by $\deg(v)$, is $|N_G(v)|$. Let $\Delta(G)$ be the maximum degree of a vertex of $G$.

The crossing number of a graph $G$, denoted by $cr(G)$, is the minimum number of crossings in a drawing of $G$ in the plane.

A drawing of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such
that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A drawing is convex if in addition the vertices are in convex position. A crossing is a point of intersection between two edges (other than a common endpoint). A drawing with no crossings is crossing-free. A graph is planar if it has a crossing-free drawing, see [4, 12, 22] for surveys. For example look at Figure 1, (planar graph $K_4$, and non-planar graphs $K_5$, $K_{3,3}$).

![Figure 1. Figures of $K_4$, $K_5$ and $K_{3,3}$ on the plan](image-url)

The crossing number is an important measure of the non-planarity of a graph [18]. Computing the crossing number is NP-hard [5], and remains so for simple cubic graphs [9, 13]. Moreover, the exact or even asymptotic crossing number is not known for specific graph families, such as complete graphs [14], complete bipartite graphs [11, 14, 16] and Cartesian products [1, 2, 6-8, 10, 15, 16, 19-21, 23, 24].

Determining the crossing number of the complete bipartite graph is one of the oldest crossing number open problems. It was first posed by Turán and known as Turán’s brick factory problem. In 1954, Zarankiewicz conjectured [24] that it is equal to

$$cr(K_{n;m}) \leq Z(m, n) = \left[\frac{m}{2}\right] \left[\frac{m-1}{2}\right] \left[\frac{n}{2}\right] \left[\frac{n-1}{2}\right].$$

He even gave a proof and a drawing that matches the lower bound, but the proof was shown to be flawed by Richard Guy [7]. Then in 1970 D.J. Kleitman proved that Zarankiewicz conjecture holds for $Min(m; n) \leq 6$ [10]. In 1993 D.R. Woodall proved it for $m \leq 8$; $n \leq 10$ [23]. Previously the best known lower bound in the general case for all $m, n \in \mathbb{N}$ was the one proved by D.J. Kleitman [10]:

$$cr(K_{n;m}) \geq \frac{1}{5} (m(m-1)) \left[\frac{n}{2}\right] \left[\frac{n-1}{2}\right].$$

Now, we have the better lower bound [11]

$$cr(K_{n;m}) \geq \frac{1}{5} (m(m-1)) \left[\frac{n}{2}\right] \left[\frac{n-1}{2}\right] + 9.9 \times 10^{-6} m^2 n^2.$$

for sufficiently large $m$ and $n$. 

References:

[4, 12, 22] for surveys.
Upper bounds on the crossing number of general families of graphs have been less studied. Obviously $cr(K_{n,m}) \leq \binom{|E(G)|}{2}$ for every graph $G$.

2. Drawing of complete Bipartite graph $K_{9,9}$

D.R. Woodall [10] used an elaborate computer search to show that Zarankiewicz conjecture holds for $K_{7,7}$ and $K_{7,9}$. Thus, one of the smallest unsettled instance of Zarankiewicz conjecture is $K_{9,9}$. For further research see paper series [8, 10, 11, 17-21].

So, we focus on the best drawing of complete bipartite graph $K_{9,9}$ and compute its crossing number for this drawing. In continue, we claim that this drawing is a best drawing for $K_{9,9}$ and $cr_D(K_{9,9})$ hold Zarankiewicz conjecture. By according the Figure 5. Also we show that by similar drawing for $K_{7,7}$ which is a best drawing of it and hold Zarankiewicz conjecture, it is maybe another proof of $cr_D(K_{7,7})$.

Before beginning present of this drawing, we give some definitions that will be used throughout the paper.

**Definition 1.** The crossing number $cr(G)$ of a graph $G$ is the smallest crossing number of any drawing of $G$ in the plane, where the crossing number $cr$ of a drawing $D$ is the number of non-adjacent edges that have a crossing in the drawing.

**Definition 2.** A good drawing a graph $G$ is a drawing where the edges are non-self-intersecting where each two edges have at most one point in common, which is either a common end vertex or a crossing.

Clearly a drawing with minimum crossing number must be a good drawing (or for strongly a best drawing) and obviously a good drawing of planar graph $G$ is the crossing-free drawing.

**Definition 3.** Suppose $V = \{v_1, v_2, ..., v_n\}$ is the vertex set of an arbitrary graph $G$. Then $E(G)$ (the edge set of $G$) is consist of $e_{i,j}$, such that $v_i$ is adjacent with $v_j$ ($\forall i, j \in \mathbb{Z}_n = \{1, 2, ..., n\}$). Now, Pair-Cross Matrix of $G$ ($CR(G) = [cr_{i,j}]_{i,j \in \mathbb{Z}_n}$) presents the number of all cross on the edge $e_{i,j}$.

It’s obvious that, if $v_i, v_j$ be the non-adjacent vertices, then $cr_{i,j} = 0$. Since, there exist many different drawing for a graph $G$, therefore we have a Pair-Cross Matrix $CR_D(G)$ for any drawing $D$ of $G$. Also, it’s obvious that all Pair-Cross Matrix $CR_D(G)$ are symmetric and the members on the original diameter are equal to zero.
Example 1. By according to Figure 1, we see that the drawing \( D_1 \) is the crossing-free drawing of \( K_4 \). So Pair-Cross Matrix of \( K_4 \) will be equal to 
\[
CR_{D_1}(K_4) = 0
\]
and also
\[
CR_{D_2}(K_4) = \begin{bmatrix}
v_1 & 0 & 0 & 0 & 1 \\
v_2 & 0 & 0 & 1 & 0 \\
v_3 & 0 & 1 & 0 & 0 \\
v_4 & 1 & 0 & 0 & 0 \\
\end{bmatrix}_{4 \times 4}
\]
(2)

Example 2. Similar Above (see Figure 1), Pair-Cross Matrix of \( K_5, K_{3,3} \) on the best drawing \( D \) will be equal to 
\[
CR_{D_3}(K_5) = \begin{bmatrix}
v_1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 1 \\
v_3 & 0 & 1 & 0 & 0 \\
v_4 & 0 & 0 & 1 & 0 \\
v_5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{5 \times 5}
\]
(3)

and
\[
CR_{D_4}(K_{3,3}) = \begin{bmatrix}
v_1 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 1 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 1 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 \\
u_1 & 0 & 0 & 0 & 0 & 0 \\
u_2 & 0 & 0 & 0 & 0 & 0 \\
u_3 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{6 \times 6}
\]
(4)
Corollary 1. The summation of all members of $CR_D(G)$ implies that is equal to the crossing number $CR_D(G)$ of a graph $G$ on the drawing $D$. In other words

$$CR_D(G) = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} cr_{i,j} = \frac{\sum_{i=1}^{n} cr_{i}}{4}$$

Definition 4. Let $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{u_1, u_2, ..., u_m\}$ be two partitions of $V(K_{m,n})$, where $V(K_{m,n})$ is the vertex set of the complete bipartite graph $K_{m,n}$. Now, the Pair-Cross Matrix $CR^*_D(K_{m,n})$ presents the number of all cross on the edge $e_{i,j} = v_i u_j$ as follow:

$$CR^*_D(K_{m,n}) = V_1 \{ [cr_{v_i u_j}]_{n \times m} \}$$

We redefine this matrix for $K_{m,n}$, because by rewrite Definition 3 for $G = K_{m,n}$ then

$$CR_D(K_{m,n}) = V_1 \rightarrow \begin{bmatrix} 0 & CR^*_D(K_{m,n})^t \cr CR^*_D(K_{m,n}) & 0 \end{bmatrix}_{(m+n) \times (m+n)} V_2$$

and $CR^*_D(K_{m,n}) = CR^*_D(K_{m,n})^t$.

Example 3. By according to Figure 1, it is obvious that modified Pair-Cross Matrix of $K_{3,3}$ is

$$CR^*_D(K_{3,3}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Corollary 2. The summation of all members of $CR^*_D(K_{m,n})$ is equal to the crossing number $CR_D(K_{m,n})$ of a complete bipartite graph on the drawing $D$. Thus

$$CR_D(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} cr^*_i j.$$
cross on the correlate edge. Notice that this change must n’t many increase other members of \( CR_D(K_{9,9}) \). By repeat this process several times, upshot we will have a good drawing of complete bipartite graph \( K_{9,9} \). See Figure 3 (For look Figure 3, attention to Appendix 1.) and Pair-Cross Matrix \( CR_D(K_{9,9}) \) is equal to

\[
CR_D(K_{9,9}) = \begin{pmatrix}
   u_1 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\
   u_2 & 16 & 12 & 8 & 4 & 0 & 0 & 4 & 8 & 12 \\
   u_3 & 12 & 9 & 6 & 3 & 0 & 0 & 3 & 6 & 9 \\
   u_4 & 8 & 6 & 5 & 4 & 3 & 4 & 5 & 6 & 7 \\
   u_5 & 4 & 3 & 4 & 5 & 6 & 8 & 7 & 6 & 5 \\
   u_6 & 0 & 0 & 3 & 6 & 9 & 12 & 9 & 6 & 3 \\
   u_7 & 0 & 0 & 4 & 8 & 12 & 16 & 12 & 8 & 4 \\
   u_8 & 4 & 3 & 5 & 7 & 9 & 12 & 10 & 8 & 6 \\
   u_9 & 8 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 8 \\
\end{pmatrix}
\]

\[
\rightarrow 64 = 4(4)^2 \\
\rightarrow 48 = 3(4)^2 \\
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\]

By refer to Figure 3 of complete bipartite graph \( K_{9,9} \), it is obvious to see that this figure is symmetric (near to symmetric) and the vertices are in the two opponent cycles (eight vertices as a common set in one of cycles and one remaining vertex is a center of another cycle). As well as, these two cycles and their covered vertices have stated in a mirror (See close up view of \( K_{9,9} \) in Figure 2).

![Figure 2. The close up view of \( K_{9,9} \) with two cycles that covered vertices (black and red cycles).](image)

Now, by according to the matrix \( CR_D(K_{9,9}) \) and Figure 3, if we redraw an edge \( e_{uv} \), then we increase the crossing number \( cr^*_{uv} \) obviously. But, an important point is number 4 and its multiples in the matrix \( CR_D(K_{9,9}) \). Number
4 is important, since \(4 = \left\lfloor \frac{9}{2} \right\rfloor\). On the other hand, number 3 is important in the matrix \(CR_D(K_{7,7})\), since \(3 = \left\lfloor \frac{7}{2} \right\rfloor\) similarly. See complete bipartite graph \(K_{7,7}\) in Figure 4 (on Appendix 2) and Pair-Cross Matrix \(CR_D(K_{7,7})\) as follow:

\[
CR_D(K_{7,7}) =\begin{pmatrix}
    u_1 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
    9 & 6 & 3 & 0 & 0 & 3 & 6 \\
    6 & 4 & 2 & 0 & 0 & 2 & 4 \\
    3 & 2 & 2 & 2 & 3 & 3 & 3 \\
    0 & 0 & 2 & 4 & 6 & 4 & 2 \\
    0 & 0 & 3 & 6 & 9 & 6 & 3 \\
    3 & 2 & 3 & 4 & 6 & 5 & 4 \\
    6 & 4 & 3 & 2 & 3 & 4 & 5
\end{pmatrix}
\rightarrow \begin{align*}
    27 &= 3(3)^2 \\
    18 &= 2(3)^2 \\
    18 &= 2(3)^2 \\
    27 &= 3(3)^2 \\
    27 &= 3(3)^2 \\
    27 &= 3(3)^2 \\
    \sum (\frac{162}{2}) &= 81 = (3)^2(3)^2
\end{align*}

3. Conclusions

In this report, we drawing \(K_{9,9}\) in the plan with 256 crossing number. We obtained this drawing by draw \(K_{9,9}\) step to step, such that we choose a large \(Cr\) on the Pair-Cross Matrix and redraw it for decrease crossing number. In fact, this work is quite tentative and experience, in other words, is handwork. In other way, we can drawing \(K_{9,9}\) by add two vertices to a best drawing \(K_{8,8}\) (Readers know that this graph have 144 crossing points in best drawing or \(Cr(K_{8,8}) = 144\)), and also we can obtain a best drawing \(K_{8,8}\) by add two vertices to best drawing \(K_{7,7}\) (\(Cr(K_{7,7}) = 81\)). In other words, For \(h = 3, ..., 9\); we can draw all complete graphs \(K_{h,h}\), that the crossing number of them hold Zarankiewicz conjecture.

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