Cesaro means in variable dyadic Hardy spaces

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Schipp, Wade, Simon and Pál [2] proved that if $f \in L_p$ ($1 < p < \infty$), then $\lim_{n \to \infty} s_n f = f$ in the $L_p$-norm, where $s_n f$ denotes the $n$-th partial sum of the Walsh-Fourier series. Jiao, Zhou, Weisz and Wu [1] generalized this result for $L_{p(\cdot)}$: if $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_{-} := \text{ess inf}_{x \in \Omega} p(x) \leq p_{+} := \text{ess sup}_{x \in \Omega} < \infty$ and for all atoms $A$, the exponent function $p(\cdot)$ satisfies that

$$\mathcal{P}(A)^{p_{-}(A)-p_{+}(A)} \leq K_{p(\cdot)},$$

(1)

then for all $f \in L_{p(\cdot)}$, $\lim_{n \to \infty} s_n f = f$ in the $L_{p(\cdot)}$-norm. Unfortunately, these results are not true if $p \leq 1$ or if $p_{-} \leq 1$. Although, for $p \leq 1$, or $p_{-} \leq 1$, we can prove convergence results with the help of Cesaro means. For $\alpha > 0$ and $n \in \mathbb{N}$, the Cesaro means of the martingale $f$ is defined by

$$\sigma_{n}^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k}^\alpha \hat{f}(k) w_k,$$

where $A_{k}^\alpha$ denotes the binomial coefficient $\binom{k+\alpha}{k}$.

We consider three types of maximal operators $(T^{(\alpha)}, U^{(\alpha)}$ and $V^{(\alpha)})$ and we prove that (under some conditions) each maximal operator is bounded from the classical dyadic martingale Hardy space $H_p$ to the classical Lebesgue space $L_p$ and these maximal operators are bounded on $L_{p(\cdot)}$.

**Theorem 1** Let $\alpha \in (0, 1]$ and $t, r > 0$ such that $a t < r/(r-t) < (1 + \alpha) t$.

1. If $0 < p \leq \infty$, then for all $f \in H_p$,

$$\left\| T^{(\alpha)} f \right\|_{p(\cdot)} \leq C_{p} \left\| f \right\|_{H_p} = \left\| U^{(\alpha)} f \right\|_{p(\cdot)} \leq C_{p} \left\| f \right\|_{H_p} = \left\| V^{(\alpha)} f \right\|_{p(\cdot)} \leq C_{p} \left\| f \right\|_{H_p}.$$

2. If $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_{-} \leq p_{+} < \infty$ and $p(\cdot)$ satisfies the condition (1), then for all $f \in L_{p(\cdot)}$,

$$\left\| T^{(\alpha)} f \right\|_{p(\cdot)} \leq C_{p} \left\| f \right\|_{p(\cdot)} = \left\| U^{(\alpha)} f \right\|_{p(\cdot)} \leq C_{p} \left\| f \right\|_{p(\cdot)} = \left\| V^{(\alpha)} f \right\|_{p(\cdot)} \leq C_{p} \left\| f \right\|_{p(\cdot)}.$$

Using this, we can prove the boundedness of the Cesaro maximal operator from $H_{p(\cdot)}$ to $L_{p(\cdot)}$, where the Cesaro maximal operator is defined by $\sigma_{n}^\alpha f := \sup_{n \in \mathbb{N}} \left| \sigma_{n}^\alpha f \right|$.

**Theorem 2** Let $0 < \alpha \in (0, 1]$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1/(\alpha+1) < p_{-} < \infty$ and suppose that $p(\cdot)$ satisfies condition (1). Then

$$\left\| \sigma_{n}^\alpha f \right\|_{p(\cdot)} \leq C \left\| f \right\|_{H_{p(\cdot)}}.$$ 

As a consequence, we can prove theorems about almost everywhere- and norm convergence.

References
