Project code: **PN-III-P4-ID-PCE-2016-0190**
Contract: **96/12.07.2017**
Project title: *Equilibrium and optimization problems: theoretical and computational approaches*
Project acronym: EQOPTIM
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SCIENTIFIC RESEARCH REPORT No. 1/2017

I. Research Team

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Assoc. Prof. Cornel Sebastian PINTEA, Ph.D.
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II.Project summary and objectives

(a) The project is mainly motivated by the growing interest in the literature regarding sensitivity, openness/metric regularity, variational principles, and practical applications concerning equilibrium and optimization problems, interest which clearly shows that these fields are appropriate for applying the modern tools of nonlinear analysis. The aim of the seven objectives of the project is to bring to light new concepts and approaches within the following specific objectives:

Obj. 1 *Parametric equilibrium problems*
Obj. 2 *Variational principles for vector equilibrium problems*
Obj. 3 *Vector optimization problems in connection with differential inclusions*
Obj. 4 *Duality and optimality conditions for set-valued optimization*
Obj. 5 *Numerical methods for equilibrium problems*
Obj. 6 *Interior-point methods for scalar optimization problems*
Obj. 7 *Algorithms for image processing and optimal location*

(b) All objectives planned for 2017 have been achieved as follows:

Obj. 1: 1 paper under review [A5 in Section III (a)]
Obj. 2: 1 accepted paper and 1 paper under review [A1 and A12 in Section III (a)]
Obj. 3: 4 papers under review [A2, A5, A7 and A8 in Section III (a)] and 2 papers in preparation*
Obj. 4: 1 paper under review [A6 in Section III (a)] and 1 paper in preparation*
Obj. 5: 1 paper under review [A4 in Section III (a)]
Obj. 6: 1 published paper and 1 paper under review [A2 and A11 in Section III (a)]
Obj. 7: 1 paper under review [A9 in Section III (a)] and 2 papers in preparation*

*see Section III (b)

III. Scientific results

(a) Published/submitted papers

12 papers have been completed in 2017:

- **1** article **has been published (online) in an ISI journal** (A2 in the next table);
- **1** article as been **accepted for publication in an ISI journal** (A1 in the next table);
- **10** papers **have been submitted to ISI journals** (A3-A12 in the next table).

Ref.	Article	Obiectives
A1	César Gutiérrez, **Gábor Kassay**, Vicente Novo, Juan-Luis Ródenas-Pedregosa: *Ekeland variational principles in vector equilibrium problem*, **SIAM Journal on Optimization**, lucrare acceptată, [JCR Science Edition 2016 IF: 1,968]	Obj. 2
A2	**Zsolt Darvay**, Petra-Renáta Takács: *New method for determining search directions for interior-point algorithms in linear optimization*, **Optimization Letters**, First Online: 20 July 2017, DOI 10.1007/s11590-017-1171-4 [JCR Science Edition 2016 IF: 1,310]	Obj. 6
A3	Monica Bianchi, **Gábor Kassay**, Rita Pini: *On a sufficient condition for weak sharp effciency in multiobjective optimization*, lucrare aflată în evaluare la revista **Journal of Optimization Theory and Applications**	Obj. 3
A4	**Gábor Kassay**, Trinh Ngoc Hai, Nguyen The Vinh: *Coupling Popov's algorithm with subgradient extragradient method for solving equilibrium problems*, lucrare aflată în evaluare la revista **Annals of Operations Research**	Obj. 5
A5	Suliman Al-Homidan, Qamrul Hasan Ansari, **Gábor Kassay**: *On sensitivity of vector equilibria by means of the diagonal subdifferential operator*, lucrare aflată în evaluare la revista **Optimization Letters**	Obj. 1 and Obj. 3
A6	Kazuki Seto, Daishi Kuroiwa, **Nicolae Popovici**: *A systematization of convexity and quasiconvexity concepts for set-valued maps, defined by l-type and u-type preorder relations*, lucrare aflată în evaluare la revista **Optimization**	Obj. 4

A7	Christian Gunther, **Nicolae Popovici**: *Jahn-Graef-Younes type algorithms for discrete vector optimization based on cone-monotone sorting functions*, lucrare aflată în evaluare la revista **Optimization**	Obj. 3
A8	Ovidiu Bagdasar, **Nicolae Popovici**: *Unifying local-global type properties in vector optimization*, lucrare aflată în evaluare la revista **Journal of Gobal Optimization**	Obj. 3
A9	**Cornel Pintea**: *Convex decompositions of convex open sets with polytopes or finite sets removed*, lucrare aflată în evaluare la revista **Indiana University Mathematics Journal**	Obj. 7
A10	Radu-Ioan Boţ, Robert Ernö Csetnek, **Szilárd Csaba László**: *Approaching nonsmooth nonconvex minimization through second order proximal-gradient dynamical systems*, lucrare aflată în evaluare la revista **Nonlinearity**	Obj. 7
A11	Petra- Renáta Takács, **Zsolt Darva**y: *A primal-dual interior-point algorithm for symmetric optimization based on a new method for finding search directions*, lucrare aflată în evaluare la revista **Optimization**	Obj. 6
A12	**Livia Mihaela Berchesan**: *Cyclically antimonotone vector equilibrium problems*, lucrare aflată în evaluare la revista **Journal of Mathematical Analysis and Applications**	Obj. 2

The main results obtained in these twelve papers are described below.

- It is well-known that Ekeland's Variational Principle (EVP, for short) represents one of the most important result of the nonlinear analysis, having various theoretical and practical applications. In the paper [***Ekeland variational principles in vector equilibrium problem***, SIAM Journal on Optimization, accepted], the **project leader, Gábor Kassay**, jointly with César Gutiérrez (University of Valladolid, Spain), Vicente Novo and Juan-Luis Ródenas-Pedregosa (Department of Applied Mathematics, UNED, Madrid, Spain), study EVP for vector equilibrium problems, where the outcome linear space is not necessarily endowed with a particular topology. Within this general framework, the authors prove different Ekeland-type variational principles, in exact and approximate form. These new results are based on a strict fixed point principle for set-valued maps, from which the authors deduce different forms of EVP, by means of algebraic notions, as well as a concept of approximate solution related to free disposal sets. The authors show that their results improve a series of other results on EVP, recently appeared in the literature and, moreover, they clarify the role of certain assumptions imposed on the vector bifunction involved in the EVP for vector equilibrium problems.

- In the paper [***New method for determining search directions for interior-point algorithms in linear optimization***, Optimization Letters, First Online: 20 July 2017, DOI 10.1007/s11590-017-1171-4] **Zsolt Darvay** together with Petra-Renáta Takács (Babeș-Bolyai University, Cluj-Napoca) present a new technique for finding search directions, which can be applied to primal-dual interior-point methods for linear optimization. They observed that the centering equation can be transformed equivalently in the following form: the square of the variance vector is equal to the variance vector. Moreover, they applied the same function to both sides of this equation and they used Newton's method, which provided new directions. They also proved that the new direction cannot be obtained by utilizing standard barriers. Some numerical results show also the practical efficiency of the algorithm.

- The **project leader** together with Monica Bianchi (University Cattolica del Sacro Cuore, Milan, Italy) and Rita Pini (University Milano-Bicocca, Italy) have established sufficient conditions for the existence of weak sharp efficient solutions of multicriteria optimization problems in their recent paper [***On a sufficient condition for weak sharp effciency in multiobjective optimization***, under review]. The notion of weak sharp efficiency (which is more restrictive than the classical notion of weak efficiency) has been introduced by Bednarczuk in 2004, by extending a similar notion proposed by Burke and Ferris in 1993 for scalar optimization problems. In order to prove the existence of weak sharp efficient solutions , the **project leader** and his co-authors propose an approach different from that of Bednarczuk, namely they show that the weak sharp efficient solutions of the multicriteria optimization problem coincides with the set of solutions of a special equilibrium problem. Then, these solutions are characterized by means of certain regularity conditions of the diagonal subdifferential operator associated to the specific equilibrium problem.

- Starting from three recent papers, by Censor and his co-authors [*The subgradient extragradient method for solving variational inequalities in Hilbert space,* J. Optim. Theory Appl. 148, 318-335 (2011)], by Malitsky and Semenov [*An extragradient algorithm for monotone variational inequalities*, Cybern. Syst. Anal. 50 (2014), 271-277], and by Lyashko and Semenov [*A new two-step proximal algorithm of solving the problem of equilibrium programming*, In: *Optimization and Applications in Control and Data Sciences* (ed. B.Goldengorin), Springer Optimization and Its Applications, Vol. 115 (2016), 315-326], the **project leader** jointly with Trinh Ngoc Hai (Hanoi University of Science and Technology, Vietnam) and Nguyen The Vinh (University of Transport and Communications, Hanoi, Vietnam) have proposed a new algorithm for solving equilibrium problems in Hilbert spaces. In their paper [***Coupling Popov's algorithm with subgradient extragradient method for solving equilibrium problems***, under review] they prove the weak convergence of the proposed algorithm and, under additional assumptions, the strong convergence toward a solution of the equilibrium problem. This algorithm improves the algorithm of Lyashko and Semenov, not only from computational point of view, but also by weakening the hypotheses of their main result. The proposed algorithm is compared with other algorithms known in the literature, its effectiveness being illustrated by numerical examples.

- The equilibrium problem in vectorial form in particular contains a series of problems from the nonlinear analysis with practical applications various domains, as for instance: vector variational inequalities, vector optimization problems, saddle points, etc. In the last two decades, several types of vector equilibrium problems have been considered, according to different types of solutions, namely strongly efficient, weakly efficient, properly efficient solutions, etc. A special attention has been devoted to equilibrium problems depending on a parameter, mainly in what concerns the stability and sensitivity of the solutions with respect to the parameter. From both theoretical and computational points of view, it is important to identify some regularity properties of the solution map, i.e., the set-valued map that associates to each value of the parameter the corresponding set of solutions. On the other hand, the notion of subdifferential for vector functions originates from the study of vector optimization problems. Based on these facts, in the paper [***On sensitivity of vector equilibria by means of the diagonal subdifferential operator***, under review] the **project leader** together with Suliman Al-Homidan (King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia) and Qamrul Hasan Ansari (Aligarh Muslim University, Aligarh, India, and King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia) have introduced the so-called diagonal subdifferential operator for vector-valued bifunctions and they have studied the regularity properties of this operator. This properties allowed the authors to prove the Lipschitz continuity of the solution map of a vector equilibrium problem with respect to the Hausdorff distance.

- In the paper [***A systematization of convexity and quasiconvexity concepts for set-valued maps, defined by l-type and u-type preorder relations***, under review] **Nicolae Popovici** jointly with Kazuki Seto and Daishi Kuroiwa (Shimane University, Matsue, Japan) give am exhaustive study of 18 classes of generalized convex set-valued maps, defined by l-type and u-type preorder relations (for comparing sets) induced by a convex cone *C* of a linear space. The authors identify those classes of *C*-quasiconvex maps that can be used to establish Crouzeix-type characterizations of *C*-convex set-valued maps by the *C*-quasiconvexity of their affine perturbations (in the sense of Ghorokhovik [*Representations of affine multifunctions by affine selections*, Set-Valued Anal. 16 (2008), 185-198]). In particular, the authors extend and refine the main results previously obtained by **Nicolae Popovici,** Daishi Kuroiwa and Matteo Rocca (University of Insubria, Varese, Italy) in their paper [*A characterization of cone-convexity for set-valued functions by cone-quasiconvexity*, Set-Valued Var. Anal. 23 (2015), 295--304].

- An effective method for computing all minimal points of a finite set with respect to a convex cone in a real liniar space is the well-known "Jahn-Graef-Younes method", presented by Jahn in his monograph [*Vector Optimization - Theory, Applications, and Extensions*, Springer, 2011, Algorithm 12.20]. This method consists in two phases: the forward iteration, proposed by Younes in [*Studies on discrete vector optimization*, PhD dissertation, University of Demiatta, Egypt, 1993] and implemented by Graef, as mentioned by Jahn in the above-mentioned monograph (p. 349), and the backward iteration, propoed by Jahn. In the work [***Jahn-Graef-Younes type algorithms for discrete vector optimization based on cone-monotone sorting functions***, under review] **Nicolae Popovici** together with Christian Günther (Martin Luther University, Halle-Wittenberg, Germany) develop two new methods, obtained by modifying the Jahn-Graef-Younes method. The first method uses a sorting procedure before the forward iteration. The second method uses a sorting procedure after the forward iteration. Examples of linear or nonlinear sorting functions are presented along with implementations of the proposed algorithms in MATLAB.

- **Nicolae Popovici** and Ovidiu Bagdasar (University of Derby, UK) have established a series of „local min – global min" and „local max – global max"properties for unified vector optimization problems, in the sense of Flores-Bazán and Hernández [*A unified vector optimization problem: complete scalarizations and applications*, Optimization 60 (2011), 1399—1419], where the notions of ideal-, strong- and weak- optimality (efficiency) are defined by means of an arbitrary set (not necessarily a convex cone). This general approach allows the authors to extend several results known in the literature, among which the „local min – global min" sproperties established by Jahn and Sachs [*Generalized quasiconvex mappings and vector optimization*, SIAM J. Control Optim. 24 (1986), 306-322] and, more recently, by Flores-Bazán and Hernández [*Optimality conditions for a unified vector optimization problem with not necessarily preordering relations*, J. Global Optim. 56 (2013), 299-315], as well as some „local max–global min" properties, established by **Nicolae Popovici** and Ovidiu Bagdasar in their paper [*Extremal properties of generalized convex vector functions*, J. Nonlinear Convex Anal., accepted, August 2015].

- In the paper [***Convex decompositions of convex open sets with polytopes or finite sets removed***, under review] **Cornel Pintea** provides convex decompositions for the convex open sets with polytopes or finite sets removed, some of which are minimal in a certain sense. The valence of a function whose restrictions to all convex subsets of its source space are injective, cannot exceed the number of convex components of such decompositions. It is therefore worth to investigate the smallest number of convex subsets of an open set needed to cover it.

- In the work [***Approaching nonsmooth nonconvex minimization through second order proximal-gradient dynamical systems***, under review] **Szilárd László** together with Radu Ioan Boț and Ernö Robert Csetnek (University of Vienna, Austria), investigate the asymptotic properties of

the trajectories generated by a second-order dynamical system of proximal-gradient type stated in connection with the minimization of the sum of a nonsmooth convex and a nonconvex smooth function. The convergence of the generated trajectory to a critical point of the objective is ensured provided a regularization of the objective function satisfies the Kurdyka- Lojasiewicz property. Convergence rates for the trajectory formulated in terms of the Lojasiewicz exponent are also provided.

- In the paper [***A primal-dual interior-point algorithm for symmetric optimization based on a new method for finding search directions***, under review] **Zsolt Darvay** jointly with Petra-Renáta Takács have generalized the approach from their previous paper [*New method for determining search directions for interior-point algorithms in linear optimization*, Optimization Letters, First Online: 20 July 2017, DOI 10.1007/s11590-017-1171-4] to symmetric cone programming.

- Based on the recent work of M. Castellani și M. Giuli [*Ekeland's principle for cyclically antimonotone equilibrium problems*, Nonlinear Analysis: Real World Applications, 32 (2016) 213-228], **Livia Mihaela Bercheșan (Miholca)** (Tehnical University of Cluj-Napoca) extends the notion of cyclic antimonotonicity (known for scalar bifunctions) to the vector case in her paper [***Cyclically antimonotone vector equilibrium problems***, under review]. In order to obtain some results on Ekeland's variational principle for vector equilibrium problems, the author characterizes the cyclic antimonotonicity in terms of a suitable approximation from below of the vector bifunction, which allows her to avoid the demanding triangle inequality property, usually required in the literature, when dealing with Ekeland's variational principle for bifunctions.

(b) Work in progress

- While visiting the Martin Luther University of Halle-Wittenberg, Germany, in the period of September 4-10, 2017, **Nicolae Popovici** has obtained together with Christian Günther some new results that may lead to a novel algorithm for computing the convex hull of compact sets in finite-dimensional linear spaces (within the objective Obj. 3).

- During a research visit at University of Vienna, Austria, in the period of October 26 – November 5, 2017, **Nicolae Popovici** has continued his research on the topic of article [A6], mentioned in section III (a). Together with Radu Ioan Boț and Ernö Robert Csetnek, he obtained new characterizations of cone-convex vector-valued functions in terms of cone-quasiconvexity of their affine perturbations (within the objective Obj. 4).

- Jointly with Val Lowndes, Stuart Berry, Chris Parkes and Ovidiu Bagdasar (University of Derby, UK), **Nicolae Popovici** contributed to Chapter 7 [*Further Use of Heuristic Methods*, pp. 199-235] of the book [Berry, S., Lowndes, V., Trovati, M. (Eds.): *Guide to Computational Modelling for Decision Processes: Theory, Algorithms, Techniques and Applications*, Springer, 2017]. While visiting the University of Derby during the period of November 17 – 26, 2017, **Nicolae Popovici** has obtained, together with Stuart Berry and Ovidiu Bagdasar, new numerical experiments relevant for traffic equilibrium problems (within the obiective Obj. 7).

- During the priod of November 17-29, 2017, while visiting the Autonomous University of Barcelona, Spain, **Cornel Pintea** has obtained, together with Juan-Enrique Martínez-Legaz, new characterizations of the closed convex sets through the range of the associated Gauss map, by paying a special attention to the Motzkin decomposable sets and the closed convex sets with compact proper faces (within the objective Obj. 7).

- Ernö Robert Csetnek (University of Vienna, Austria) has visited the Babeș-Bolyai University of Cluj-Napoca during October 2017, as a STAR-UBB Fellow. During this period, **Anca Grad** has continued her collaboration with Ernö Robert Csetnek and Radu Ioan Boț, in order to develop a new inertial-type algorithm for solving monotone inclusion problems involving the sum of three terms (within the objective Obj. 3).

IV. Dissemination of research results

The scientific results mentioned within Section III of this report have been presented by the authors (members of the project research team) at **15** conferences, workshops and research seminars in Romania or abroad, namely:

- **6 conferences and workshops abroad [**CS1, CS2, CS6, CS7, CS13 and CS16 in the next table**];**
- **5 research seminars abroad [**CS3, CS5, CS9, CS10 and CS14 in the next table**];**
- **2 conferences in Romania [**CS4 and CS11 in the next table**];**
- **3 research seminars in Romania [**CS8, CS12 and CS15 in the next table**].**

Ref.	**Conference/workshop /research seminar**
CS1	**Szilárd Csaba László**: *Second order dynamical systems with penalty terms associated to monotone inclusions*, GPCO 2017- 7th German-Polish Conference on Optimization, Bedlewo, Poland, Augut 27 – September 1, 2017, https://www.impan.pl/konferencje/bcc/2017/17-gpco/preliminary_program_18_08_2017ms_4.pdf
CS2	**Cornel Pintea**: Generalized monotone operators on finite dimensional spaces. Applications, XII International Symposium on Generalized Convexity and Monotonicity, Hajdúszoboszló, Hungary, August 27 - September 2, 2017, http://gcm.up.krakow.pl/
CS3	**Nicolae Popovici**: *Local-global type properties for generalized convex vector functions*, Oberseminar Optimierung, Martin Luther University of Halle, Germany, September 7, 2017, http://optimierung.mathematik.uni-halle.de/veranstaltungen/oberseminaroptimierung/
CS4	**Anca Grad**: *Monotone inclusions through inertial algorithms*, 6th International Conference on Mathematics and Informatics, Targu Mures, România, September 7–9, 2017, http://mitis.ro/mathinfo/2017/
CS5	**Zsolt Darvay**: *Interior-point algorithms from the perspective of neighborhoods and directions*, Research Seminar, Department of Differential Equations, Budapest University of Technology and Economics, Budapest, Hungary, September 21, 2017.
CS6	**Nicolae Popovici** (**chair of contributed talks session** Vector-valued optimization and related issues): *Local-global type properties for generalized convex vector functions*, 18th French – German – Italian Conference on Optimization, Paderborn, Germany, September 25 - 28, 2017, https://math.uni-paderborn.de/ag/mathematik-und-ihre-anwendungen/fgi-2017/programme/contributed-talks-on-tuesday/

CS7	Petra Renáta Rigó, **Zsolt Darvay**, Tibor Illés: *Infeasible interior-point algorithms for linear optimization problems*, The 14th International Symposium on Operations Research in Slovenia - SOR'17, Bled, Slovenia, September 27-29, 2017, http://sor17.fov.uni-mb.si/
CS8	**Nicolae Popovici**: *Unifying local-global type properties in vector optimization*, Seminar of the Research Group on Analysis and Optimization, Babes-Bolyai University, Cluj-Napoca, Romania, October 5, 2017, http://www.cs.ubbcluj.ro/~grupanopt/
CS9	**Gábor Kassay**: *Existence results for vector equilibrium problems given by a sum of two functions*, Research Seminar, Department of Applied Mathematics, University of Granada, Spain, October 25, 2017.
CS10	**Gábor Kassay**: *Stability of equilibria via regularity of the diagonal subdifferential operators*, Research Seminar, Department of Applied Mathematics, University of Granada, Spain, October 27, 2017
CS11	Petra- Renáta Takács, **Zsolt Darvay**: *New algebraic transformation for determination of interior-point method for linear programming*, 8th Mathematics and Computer Science with Applications Conference, Cluj-Napoca, Romania, November 3-5, 2017, http://www.cs.ubbcluj.ro/~darvay/eme/mtne2017/
CS12	Livia Mihaela Berchesan: *Cyclically antimonotone vector equilibrium problems*, Seminar of the Research Group on Analysis and Optimization, Babes-Bolyai University, Cluj-Napoca, Romania, November 9, 2017, http://www.cs.ubbcluj.ro/~grupanopt/
CS13	**Gábor Kassay** (**guest of honor and plenary speaker**): *On a suffcient condition for weak sharp effciency in multiobjective optimization*, International Conference on Analysis and its Applications, Aligarh Muslim University, Aligarh, India, Novermber 20 - 22, 2017, https://icaa2017.com/ , https://icaa2017.com/plenaryspeakers
CS14	**Nicolae Popovici**: *New algorithms for solving discrete vector optimization problems*, Research Seminar, Department of Electronics, Computing & Mathematics, University of Derby, UK, November 22, 2017
CS15	**Gábor Kassay**: *On a sufficient condition for weak sharp efficiency in multiobjective optimization*, Seminar of the Research Group on Analysis and Optimization, Babes-Bolyai University, Cluj-Napoca, Romania, December 7, 2017, http://www.cs.ubbcluj.ro/~grupanopt/
CS16	**Gábor Kassay** (**organizer of the special session** Equlibrium and vector optimization problems): *Coupling Popov's algorithm with subgradient extragradient method for solving equilibrium problems*, 4th Conference on Optimization Methods and Software, Havana, Cuba, December 16-20, 2017, http://wias-berlin.de/workshops/oms2017/

Project leader,
Prof. Gábor Kassay, Ph.D.