Project code: **PN-II-ID-PCE-2011-3-0024**
Contract: **157/05.10.2011**, financed by UEFISCDI
Project title: ***The structure and sensitivity of the solution sets of variational inequalities, optimization and equilibrium problems under generalized monotonicity***
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SCIENTIFIC RESEARCH REPORT No. 6
covering the period of 01.01.2016 – 04.10.2016

I. Research Team

Prof. **Gabor KASSAY**, PhD (project leader)
Prof. **Nicolae POPOVICI**, PhD habil
Assoc. Prof. **Cornel PINTEA**, PhD
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Petra-Renáta **TAKÁCS**, PhD student

II.Project summary and objectives

(a) The project is mainly motivated by the growing literature in scalar and vector optimization problems, variational inequalities, and equilibrium problems, which neatly shows that these fields are appropriate for applying the modern tools of variational analysis. The following objectives have been proposed in the funding application:

- **O1** *Studying condition numbers and metric regularity within parametric variational inequalities and parametric equilibrium problems*
- **O2** *Identifying classes of generalized monotone operators for which local and global monotonicity are equivalent and deduce injectivity results*
- **O3** *Studying the structure of the solution sets for generalized monotone operators*
- **O4** *Characterizing the subdifferential for certain classes of generalized monotone operators*
- **O5** *Approaching the sum problem for maximal monotone operators*
- **O6** *Constructing algorithms for variational inequalities and equilibrium problems*
- **O7** *Extending the proximal point algorithm for equilibrium problems to reflexive Banach spaces*
- **O8** *Characterizing generalized convex vector functions by scalarization*
- **O9** *Studying the structure of the solution sets of vector variational inequalities and equilibrium problems*

(b) All objectives planned for the period 01.01.2016 – 04.10.2016 have been achieved as follows:

O6: 1 published paper [A1 in Section III (a)] and 1 paper under review [A3 in Section III (a)]
O8: 1 habilitation thesis* and 2 papers in preparation**
O9: 1 paper under review [A2 in Section III (a)], 1 habilitation thesis* and 1 paper in preparation**.

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*see Section III (b)
**see Section III (c)

III. Scientific results

(a) Published/submitted papers

2 papers have been completed during the period of 01.01.2016 – 04.10.2016:

- **1** article **has been published in an ISI journal** (A1 in the next table);
- **2** articles **have been submitted for publication to ISI journals** (A2 and A3 in the next table).

Ref.	Article	Obiectives
A1	Zsolt Darvay, Ingrid-Magdolna Papp, **Petra-Renáta Takács**: *Complexity analysis of a full-Newton step interior-point method for linear optimization*, **Period. Math. Hungar.**, 73 (2016) (1), 27–42, doi: 10.1007/s10998-016-0119-2 [JCR Science Edition 2015 IF: 0,286]	O6
A2	**Szilárd László**: *Minimax results on dense sets and dense families of functionals*, under review	O9
A3	**Gabor Kassay**, Tring Ngoc Hai, Nguyen The Vinh: *Coupling Popov's algorithm with subgradient extragradient method for solving equilibrium problems*, under review	O6

The main results obtained in these two papers are described below.

- In the paper [*Complexity analysis of a full-Newton step interior-point method for linear optimization*, **Period. Math. Hungar.**, 73 (2016) (1), 27-42] **P.-R. Takács** together with Zs. Darvay and I.-M. Papp (Babeș-Bolyai University, Cluj-Napoca) introduced a new primal-dual interior-point method for linear optimization which is based on a new search direction. They applied the method of algebraic equivalent transformation on the nonlinear equation of the system which defines the central path. For this, they considered a new function, namely the difference of the identity and the square root function. After that they applied Newton's method in order to obtain the new search directions. In spite of the fact that the analysis of the algorithm was more complicated in this case, they proved that the algorithm has the same complexity as the best known interior-point methods for linear optimization. This study opens the way for further extensions within objective **O6** of our project.

- Following the research direction on self-segment-dense sets, in the work [*Some minimax results on dense sets and dense families of functionals*, under review at SIAM J. Optim.] **Szilárd László** obtained some new minimax results on dense sets. The techniques used in proofs rely on the powerful tools of Convex Analysis and Functional Analysis, and is shown that the notion of a self-segment-dense set, introduced in the paper [**Sz. László**, A. Viorel: *Generalized monotone operators on dense sets*, **Numer. Funct. Anal. Optim.**, 36 (2015), 901-929], is essential in all of these results, since they cannot be replaced by usual denseness. As a matter of fact, minimax results on dense sets are absent in the literature. In this paper a motivation is given for this absence, by an example is shown that the general minimax results of Fan and Sion cannot be extended on usual dense sets. The minimax theorems obtained in this paper lead to some results, about the denseness of some parameterized families of functionals in the Banach space of bounded functions, which can be viewed as generalizations or extensions of James Compactness Theorem (one of the fundamental Theorems of Functional Analysis). This paper fits into objective **O9** of our project.

- Based on the recent works by Y. Censor, A. Gibali and S. Reich [*The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., 148 (2011), 318-335], Yu Malitsky [*Projected reflected gradient methods for monotone variational inequalities*, SIAM J. Optim., 25 (2015), 502-520] and Dang Van Hieu [*Weak and strong convergence of subgradient extragradient methods for pseudomonotone equilibrium problems*, Commun. Korean Math. Soc., accepted], the **project leader** (**G. Kassay**) together with Tring Ngoc Hai (University of Science and Technology, Hanoi, Vietnam) and Nguyen The Vinh (University of Transport and Communications, Hanoi, Vietnam) proposed a new scheme for solving pseudomonotone equilibrium problems in real Hilbert spaces. Weak and strong convergence results are suitably established. Their algorithm improves the recent one announced by Hieu not only from computational point of view. but also in some assumptions imposed on his main result. A comparative numerical study is carried out between the three algorithms (the algorithms of T. D. Quoc, L. D. Muu and N. V. Hien [*Extragradient algorithms extended to equilibrium problems*, Optimization, 57 (2008), 749-776], Ya. I. Vedel and V.V. Semenov [*New two-step proximal method for equilibrium problem*, J. Comput. Appl. Math., 118 (2015), 15-23] and Hieu) and the new one. Experimental results (using MATHLAB) show that their algorithm is more efficient than the previous ones. This study is done within objective **O6** of the project.

(b) Habilitation Thesis (related to the projecy objectives **O8** and **O9**)

Nicolae Popovici: *The role of generalized convexity in vector optimization and related variational problems*, Habilitation Thesis, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania, defended March 25, 2016, Summary available at http://doctorat.ubbcluj.ro/wp-content/uploads/2016/02/rezumat-EN.pdf

(c) Work in progress

- **Nicolae Popovici** and O. Bagdasar (University of Derby, UK) succeeded to extend the main results of their papers [*Local maximum points of explicitly quasiconvex functions*, **Optim. Lett**., 9 (2015) (4), 769–777] and [*Extremal properties of generalized convex vector functions*, **J. Nonlinear Convex Anal.**, accepted] to the framework of explicitly *K*-quasiconvex vector-valued functions in the sense of F. Flores-Bazan si E.Hernandez [*A unifed vector optimization problem: complete scalarizations and applications,* Optimization, 60 (2011), 1399-1419]. For appropriate choices of *K one obtains "local min –global min" and "local max – global min"properties, which play an important role in vector optimization* (within objectives **O8** and **O9**).

- **Cornel Pintea** together with Juan Enrique Martinez-Legaz (Universitat Autònoma de Barcelona, Spain) studied closed convex sets with Minkowski slices (within objective **O8**). They obtained a characterization of the closed convex sets whose intersections with subspaces transversal to the lineality of such a closed convex set are Minkowski sets. Furthermore, the authors plan to investigate some other properties of such sets.

IV. Dissemination of research results

The scientific results mentioned within Section III of this report have been disseminated by the authors (members of the project research team) by presenting **13** talks at conferences, workshops and research seminars in Romania or abroad, namely:

- **6 international conferences and workshops,**
- **4 research seminars abroad;**
- **3 national conference/research seminars in Romania**

The detailed list of talks is available on the project webpage at

http://www.cs.ubbcluj.ro/~grupanopt/PN-II-ID-PCE-2011-3-0024/index_eng.htm

Proiect leader,
Prof. Dr. Gabor Kassay