ORDERED VECTOR SPACES AND ELEMENTS OF CHOQUET THEORY (A COMPENDIUM)

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Contents

1. Introduction

The aim of these notes is to present a compilation of some basic results on ordered vector spaces and positive operators and functionals acting on them. A short presentation of Choquet theory is also included. They grew up from a talk I delivered at the Seminar on Analysis and Optimization. The presentation follows mainly the books [3], [9], [19], [22], [25], and [11], [23] for the Choquet theory. Note that the first two chapters of [9] contains a thorough introduction (with full proofs) to some basics results on ordered vector spaces. In spite of the time passed from their publishing, the references [19] and [22] are still valuable sources. There are a lot of good books (and probably even more bad ones) devoted to ordered vector spaces and to their applications in various fields of mathematics.

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The bibliography at the end of this paper is very selective, but Mathscinet, or ZblMATH, could help for further references. The terminology also differs from author to author, making difficult to merge results from different sources. For instance by a "cone" some authors understand a set K satisfying only $\mathbb{R}_+K \subset K$, without convexity, reserving the term convex cone for such an object which is also convex. So we must take care when applying a result - we have to check author's terminology. I chose the term "wedge" for a convex cone and "cone" for a pointed convex cone (i.e. such that $K \cap (-K) = \{0\}$). It is possible that some results formulated for "cones" hold for "wedges" too. In some books one does not pay attention to wedges and one works only with cones and orders satisfying the antisymmetry condition.

Applications are not included. All the books mentioned in the bibliography contains applications of the theory to various domains. The book [14] is devoted to applications of theory of positive linear operators to Korovkin approximation theory, while applications of Choquet theory to the same topic can be found in the book [4], and in [5], [7] to potential theory. The book [16] contains applications of ordered vector spaces to integration theory.

2. Cones in vector spaces

2.1. **Ordered vector spaces.** A *pre-order* on a set Z is a reflexive and transitive relation $\le$ on Z. If the relation $\le$ is also antisymmetric then it is called an *order* on Z.

A nonempty subset W of a real vector space is called a *wedge* if

$$\begin{aligned} &\text{(C1)} \qquad W + W \subset W, \\ &\text{(C2)} \qquad tW \subset W, \quad \text{for all } \ t \ge 0. \end{aligned} \tag{2.1}$$

The wedge W induces a pre-order on X given by

$$x \le_W y \iff y - x \in W. \tag{2.2}$$

The notation $x <_W y$ means that $x \le_W y$ and $x \ne y$.

If there no danger of confusion the subscripts will be omitted.

This pre-order is compatible with the linear structure of X, that is

$$\begin{aligned} &\text{(i)} \qquad x \le y \Longrightarrow x + z \le y + z, \quad \text{and} \\ &\text{(ii)} \qquad x \le y \Longrightarrow tx \le ty, \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$ and $t \in \mathbb{R}_+$, where $\mathbb{R}_+ = \{t \in \mathbb{R} : t \ge 0\}$. This means that one can add inequalities

$$x \le y \text{ and } x' \le y' \Longrightarrow x + x' \le y + y',$$

and multiply by positive numbers

$$x \le y \iff tx \le ty,$$

for all $x, x', y, y' \in X$ and $t > 0$. The multiplication by negative numbers reverses the inequalities

$$\forall t < 0, \ \ (x \le y \iff tx \ge ty).$$

As a consequence of this equivalence, a subset A of X is bounded above iff the set $-A$ is bounded below. Also

$$\inf A = -\sup(-A) \quad \sup A = -\inf(-A).$$

Remark 2.1. It follows that in definitions (or hypotheses) we can ask only one order condition. For instance, if we ask that every bounded above subset of an ordered vector space has a supremum, then every bounded below subset will have an infimum, and consequently, every bounded subset has an infimum and a supremum. Similarly, if a linear pre-order is upward directed, then it is automatically downward directed, too.

Also, the pre-order $\leq_W$ is total iff $X = W \cup (-W)$.

Obviously, the wedge W agrees with the set of positive elements in X,

$$W = X_+ := \{x \in X : 0 \leq_W x\}.$$

Conversely, if $\leq$ is a pre-order on a vector space X satisfying (2.3) (such an order is called a *linear pre-order*), then $W = X_+$ is a wedge in X and $\leq = \leq_W$. Consequently, there is a perfect correspondence between linear pre-orders on a vector space X and wedges in X and so any property in an ordered vector space can be formulated in terms of the pre-order or of the wedge.

A *cone* K is a wedge satisfying the condition

$$\text{(C3)} \qquad K \cap (-K) = \{0\}. \tag{2.4}$$

This is equivalent to the fact that the induced pre-order is antisymmetric,

$$x \leq y \text{ and } y \leq x \Longrightarrow y = x, \tag{2.5}$$

for all $x, y \in X$, that it is an order on X.

An *order interval* in an ordered vector space (X, K) is a (possibly empty) set of the form

$$[x; y] = \{z \in X : x \leq z \leq y\} = (x + K) \cap (y - K),$$

for some $x, y \in X$. It is clear that an order interval $[x; y]$ is a convex subset of X and that

$$[x; y] = x + [0; y - x].$$

A subset A of X is called *order convex* (or *full*, or *saturated*) if $[x; y] \subset A$ for every $x, y \in A$. Since the intersection of an arbitrary family of order convex sets is order convex, we can define the order convex hull $[A]$ of a nonempty subset A of X as the intersection of all order convex subsets of X containing A, i.e. the smallest order convex subset of X containing A. It follows that

$$[A] = \bigcup\{[x; y] : x, y \in A\} = (A + K) \cap (A - K). \tag{2.6}$$

Obviously, A is order convex iff $A = [A]$.

Remark 2.2. If $x \leq y$, then $[x; y]_a \subset [x; y]_o$, where $[x; y]_a = \{x + t(y - x) : t \in [0; 1]\}$ denotes the algebraic segment connecting x and y, and $[x; y]_o$ the order interval. Taking $X = \mathbb{R}^2$ with the coordinate order and $x = (0, 0)$, $y = (1, 1)$, then $[x; y]_o$ equals the (full) quadrat with the vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(0, 1)$, so it is larger than the segment $[x; y]_a$ which is the diagonal of the quadrat.

Note. In what follows the order segments will be denoted by $[x; y]$ and the algebraic ones by $[x; y]_a$.

We mention also the following result.

Proposition 2.3 ([9])**.** *Let $(X, \leq)$ be an ordered vector space. Then the order $\leq$ is total iff every order convex subset of X is convex.*

We shall consider now some algebraic-topological notions concerning the subsets of a vector space X. Let A be a subset of X.

The subset A is called:

- *convex* if $[x; y]_a \subset A$ for every $x, y \in A$;
- *balanced* if $\lambda A \subset A$ for every $|\lambda| \leq 1$;
- *symmetric* if $-A = A$;
- *absolutely convex* if it is convex and balanced;
- *absorbing* if $\{t > 0 : x \in tA\} \neq \emptyset$ for every $x \in X$.

The following equivalences are immediate:

$$\begin{aligned} A \text{ is absolutely convex } &\iff \forall a, b \in A,\ \forall \alpha, \beta \in \mathbb{R},\ \text{with } |\alpha| + |\beta| = 1, \quad \alpha a + \beta b \in A \\ &\iff \forall a, b \in A,\ \forall \alpha, \beta \in \mathbb{R},\ \text{with } |\alpha| + |\beta| \leq 1, \quad \alpha a + \beta b \in A. \end{aligned}$$

Notice that a balanced set is symmetric and a symmetric convex set containing 0 is balanced.
The following properties are easily seen.

Proposition 2.4. *Let X be an ordered vector space and $A \subset X$ nonempty. Then*

1. *If A is convex, then $[A]$ is also convex.*
2. *If A is balanced, then $[A]$ is also balanced.*
3. *If A is absolutely convex, then $[A]$ is also absolutely convex.*

One says that a is an *algebraic interior* point of A if

$$\forall x \in X,\ \exists \delta > 0,\ \text{s.t. } \forall \lambda \in [-\delta, \delta],\ a + \lambda x \in A. \tag{2.7}$$

The (possibly empty) set of all interior points of A, denoted by $\operatorname{aint}(A)$, is called the *algebraic interior* (or the *core*) of the set A. It is obvious that if X is a TVS, then $\operatorname{int}(A) \subset \operatorname{aint}(A)$. In finite dimension we have equality, but the inclusion can be proper if X is infinite dimensional. A cone K is called *solid* if $\operatorname{int}(K) \neq \emptyset$.

Remark 2.5. Zălinescu [26] uses the notation A^i for the algebraic interior and ${}^iA$ for the algebraic interior of A with respect to its affine hull (called the relative algebraic interior). In his definition of an algebraic interior point of A one asks that the conclusion of (2.7) holds only for $\lambda \in [0; \delta]$, a condition equivalent to (2.7).

The set A is called *linearly open* (or *algebraically open*) if $A = \operatorname{aint}(A)$, and *linearly closed* if $X \setminus A$ is linearly open. This is equivalent to the fact that any line in X meets A in a closed subset of the line. The smallest linearly closed set containing a set A is called the *lineal* (or *algebraic*) closure of A and it is denoted by $\operatorname{acl}(A)$. Again, if X is a TVS, then any closed subset of X is linearly closed. The subset A is called *linearly bounded* if the intersection with any line D in X is a bounded subset of D. Similar to the topological case one can prove that

$$a \in \operatorname{aint}(A) \text{ and } b \in A \text{ and } \lambda \in [0; 1) \implies (1 - \lambda)a + \lambda b \in \operatorname{aint}(A).$$

Now we shall consider some further properties of linear orders. A linear order $\leq$ on a vector space X is called

- *Archimedean* if for every $x, y \in X$,

$$\forall n \in \mathbb{N},\ nx \leq y \implies x \leq 0; \tag{2.8}$$

- *almost Archimedean* if for every $x, y \in X$,

$$\forall n \in \mathbb{N},\ -y \leq nx \leq y \implies x = 0; \tag{2.9}$$

- *everywhere non-Archimedean* if

$$\forall x \in X, \exists y \in K,\ \ \forall n \in \mathbb{N},\ -y \leq nx \leq y. \tag{2.10}$$

The following four propositions are taken from Breckner [9] and Jameson [19]. In all of them X will be a real vector space and $\leq$ a linear order on X given by the wedge $W = X_+$.

Proposition 2.6. *Let X be a vector space ordered by a wedge W. T.f.a.e.*

1. *The order $\leq$ is Archimedean.*
2. *The wedge W is linearly closed.*
3. *For every $x \in X$ and $y \in W$, $0 = \inf\{n^{-1}x : n \in \mathbb{N}\}$.*

4. *For every* $x \in X$ *and* $y \in W$, $nx \leq y$, *for all* $n \in \mathbb{N}$, *implies* $x \leq 0$.
5. *For every* $A \subset \mathbb{R}$ *and* $x, y \in X$, $y \leq \lambda x$ *for all* $\lambda \in A$, *implies* $y \leq \mu x$, *where* $\mu = \inf A$.

Proposition 2.7. *T.f.a.e.*

1. *The order is almost-Archimedean.*
2. $\operatorname{acl}(W)$ *is a wedge.*
3. *Every order interval in* X *is linearly bounded.*

Proposition 2.8. *T.f.a.e.*

1. *The order is everywhere non-Archimedean.*
2. $\forall x \in X,\ \exists y \in W,\ \forall \lambda > 0,\ x \leq \lambda y.$
3. $\operatorname{acl}(W) = X.$

Example 2.9. Let c_{00} be the space of all real sequences having only finitely many nonzero terms. Consider the cone K formed by all $y \in c_{00}$ having the last non-zero term positive plus the null element. Then the order given by K (called the reverse lexicographic order) is everywhere non-Archimedean.

Indeed, if $x = (x_1, x_2, \ldots, x_n, 0, \ldots)$ where x_n is the last non-zero term, then $-e_{n+1} \leq nx \leq e_{n+1}$, for all $n \in \mathbb{N}$, where e_{n+1} is the sequence having 1 at the $n+1$ position and 0 in rest.

A wedge W in X is called *generating* if $X = W - W$. The order $\leq$ is called *upward* (*downward*) *directed* if for every $x, y \in X$ there is $z \in X$ such that $x \leq z$, $y \leq z$ (respectively, $x \geq z$, $y \geq z$). If the order is linear, then these two notions are equivalent, so we can say simply that $\leq$ is directed.

Proposition 2.10. *T.f.a.e.*

1. *The wedge* W *is generating.*
2. *The order* $\leq$ *is directed.*
3. $\forall x \in X,\ \exists y \in W,\ x \leq y.$

Remark 2.11. It is obvious that a linear order $\leq$ is total iff $X = W \cup (-W)$.

Let (X, W) be an ordered vector space. An element $u \in W$ is called an *order unit* if the set $[-u; u]$ is absorbing. It is obvious that an order unit must be different of 0 (provided $X \neq \{0\}$).

Proposition 2.12 ([9] and [19])**.** *Let* X *be a vector space ordered by a wedge* W *and* $u \in W \setminus \{0\}$. *T.f.a.e.*

1. *The element* u *is an order unit.*
2. *The order interval* $[0; u]$ *is absorbing.*
3. *The element* u *belongs to the algebraic interior of* W.
4. $[\mathbb{R}u] = X.$

We mention also the following results relating order units and the order.

Proposition 2.13 ([19])**.** *Let* X *be a vector space ordered by a wedge* W *and* u *an order unit in* X.

1. *The order is Archimedean iff* $x \leq \lambda u$ *for all* $\lambda > 0$ *implies* $x \leq 0$.
2. *The order is almost-Archimedean iff* $-\lambda u \leq x \leq \lambda u$ *for all* $\lambda > 0$ *implies* $x = 0$.

An ordered vector space X is called a vector lattice if any two elements $x, y \in X$ have a supremum, denoted by $x \vee y$. It follows that they have also an infimum, denoted by $x \wedge y$, and these properties extend to any finite subset of X. In terms of cones, vector lattices can be characterized as follows.

Proposition 2.14 ([3], Th. 1.26 or [22], P. 3.8)**.** *Let* X *be a vector space ordered by a cone* K. *T.f.a.e.*

1. X *is a vector lattice.*

2. *For every* $x, y \in X$ *there exists* $z \in X$ *s.t.*
$$(x+K) \cap (y+K) = z+K,$$
in which case $z = x \vee y$.
3. *For for every* $x, y \in X$ *there exists* $z \in X$ *s.t.*
$$(x-K) \cap (y-K) = z-K,$$
in which case $z = x \wedge y$.
4. *For every* $x \in X$ *there exists* $z \in X$ *s.t.*
$$(x+K) \cap K = z+K,$$
in which case $z = x \vee 0$.
5. *For for every* $x \in X$ *there exists* $z \in X$ *s.t.*
$$(x-K) \cap (-K) = z-K,$$
in which case $z = x \wedge 0$.

The following proposition collects some useful calculus rules in vector lattices (reminding some well known formulae concerning real numbers). For a vector lattice $(X, \leq)$ denote by
$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = (x) \vee (-x),$$
the positive part, the negative part and the absolute value (or the modulus) of an element $x \in X$.

Proposition 2.15. *The following calculus rules hold in a vector lattice* $(X, \leq)$.
1. $x = x^+ - x^-$, *(and so* $X = X_+ - X_+$, *that is the cone* X_+ *is generating),* $|x| = x^+ + x^-$ *and* $x^+ \wedge x^- = 0$ *(that is the elements* x^+ *and* x^- *are* disjoint*).*
2. $x \vee y = \frac{1}{2}\left[(x+y) + |x-y|\right]$ *and* $x \wedge y = \frac{1}{2}\left[(x+y) - |x-y|\right]$.
3. $|x| \vee |y| = \frac{1}{2}\left[|x+y| + |x-y|\right]$ *and* $|x| \wedge |y| = \frac{1}{2}\left[|x+y| - |x-y|\right]$.
4. $x + y = x \vee y + x \wedge y$.
5. $(\lambda x) \vee (\lambda y) = \lambda(x \vee y)$ *and* $(\lambda x) \wedge (\lambda y) = \lambda(x \wedge y)$ *for* $\lambda \geq 0$.
6. $|\lambda x| = |\lambda|\,|x|$ *for* $x \in \mathbb{R}$.
7. $|x+y| \leq |x| + |y|$ *and* $|x-y| \geq \big||x| - |y|\big|$.

The ordered vector space X is called *order complete* (or *Dedekind complete*) if every bounded from above subset of X has a supremum and *order σ-complete* (or *Dedekind σ-complete*) if every bounded from above countable subset of X has a supremum. The fact that every bounded above subset of X has a supremum is equivalent to the fact that every bounded below subset of X has an infimum. Indeed, if A is bounded above, then $\sup\{y : y \text{ is a lower bound for } A\} = \inf A$.

Remark 2.16. An ordered vector space X is order complete iff for each pair A, B of nonempty subsets of X such that $A \leq B$ there exists $z \in X$ with $A \leq z \leq B$.

This similarity with "Dedekind cuts" in $\mathbb{R}$ justifies the term Dedekind complete used by some authors. Here $A \leq B$ means that $a \leq b$ for all $(a, b) \in A \times B$.

The following results give characterizations of these properties in terms of directed subsets.

Proposition 2.17 ([3], Th. 1.20)**.** *Let* X *be a vector lattice.*
1. *The space* X *is order complete iff every upward directed bounded above subset of* X *has a supremum.*
2. *The space* X *is Dedekind σ-complete iff every upward directed bounded above countable subset of* X *has a supremum.*

2.2. **Ordered topological vector spaces (TVS).** In the case of an ordered TVS (X,τ) some connections between order and topology hold. In the following propositions (X,τ) will be a TVS with an order $\leq$ generated by a wedge W or by a convex cone K. We start by a simple result.

Proposition 2.18. *A wedge W is closed iff the inequalities are preserved by limits, meaning that for all nets $(x_i : i\in I)$, $(x_i : i \in I)$ in X,*

$$\forall i\in I,\ x_i\leq y_i \ \text{ and } \ \lim_i x_i = x,\, \lim_i y_i = y \Longrightarrow\ x\leq y.$$

Other results are contained in the following proposition.

Proposition 2.19 ([3], Lemmas 2.3 and 2.4)**.** *Let (X,τ) be a TVS ordered by a τ-closed cone K. Then*

1. *The topology τ is Hausdorff.*
2. *The cone K is Archimedean.*
3. *The order intervals are τ-closed.*
4. *If $(x_i : i\in I)$ is an increasing net which is τ-convergent to $x\in X$, then $x = \sup_i x_i$.*
5. *Conversely, if the topology τ is Hausdorff, $\operatorname{int}(K)\neq\emptyset$ and K is Archimedean, then K is τ-closed.*

Remark 2.20. Recall that a cone with nonempty interior is called *solid*.

2.3. **Normal cones in TVS and in LCS.** Now we introduce a very important notion in the theory of ordered vector spaces. A cone K in a TVS (X,τ) is called *normal* if there exists a neighborhood basis at 0 formed of order convex sets. A seminorm p on an ordered vector space X is called *monotone* if

$$\forall x,y\in K,\ \ x\leq y \implies p(x)\leq p(y),$$

or equivalently,

$$\forall x,z\in K,\ \ p(x)\leq p(x+z).$$

The following characterization are taken from [9] and [22].

Theorem 2.21. *Let (X,τ) be a TVS. T.f.a.e.*

1. *The cone K is normal.*
2. *There exists a basis $\mathcal{B}$ formed of order convex balanced 0-neighborhoods.*
3. *There exists a basis $\mathcal{B}$ formed of balanced 0-neighborhoods such that for every $B\in\mathcal{B}$, $y\in B$ and $0\leq x\leq y$ implies $x\in B$.*
4. *There exists a basis $\mathcal{B}$ formed of balanced 0-neighborhoods such that for every $B\in\mathcal{B}$, $y\in B$ implies $[0;y]\subset B$.*
5. *There exists a basis $\mathcal{B}$ formed of balanced 0-neighborhoods and a number $\gamma>0$ such that for every $B\in\mathcal{B}$, $[B]\subset\gamma B$.*
6. *If $(x_i : i\in I)$ and $(y_i : i\in I)$ are two nets in X such that $\forall i\in I,\ 0\leq x_i\leq y_i$ and $\lim_i y_i = 0$, then $\lim_i x_i = 0$.*

If further, X is a LCS, then the fact that the cone K is normal is equivalent to each of the conditions 2–6, where the term "balanced" is replaced with "absolutely convex".

Remark 2.22. Condition 6 can be replaced with the equivalent one:

If $(x_i : i\in I)$, $(y_i : i\in I)$ and $(z_i : i\in I)$ are nets in X such that $\forall i\in I,\ x_i\leq z_i\leq y_i$ and $\lim_i x_i = x = \lim_i y_i$, then $\lim_i z_i = x$.

We mention also the following result.

Proposition 2.23 ([22])**.** *If K is a normal cone in a LCS (X,τ), then every continuous linear functional on X is the difference of two positive continuous linear functionals.*

The existence of a normal solid cone in a TVS makes the topology normable.

Proposition 2.24 ([3], p. 81, Exercise 11, and [22]). *If a Hausdorff TVS (X,τ) contains a solid τ-normal cone, then the topology τ is normable.*

In order to give characterizations of normal cones in LCS we consider some properties of seminorms. Let $\gamma>0$. A seminorm p on a vector space X is called

- γ-*monotone* if $0\le x\le y \implies p(x)\le\gamma p(y)$;
- γ-*absolutely monotone* if $-y\le x\le y \implies p(x)\le\gamma p(y)$;
- γ-*normal* if $x\le z\le y \implies p(z)\le\gamma\max\{p(x),p(y)\}$.

A 1-monotone seminorm is called *monotone*. Also a seminorm for which is γ-monotone for some $\gamma>0$ is called sometimes semi-monotone (see [13]).

These properties can be characterized in terms of the Minkowski functional attached to an absorbing subset A of a vector space X, given by

$$p_A(x)=\inf\{t>0: x\in tA\},\quad (x\in X.) \tag{2.11}$$

It is well known that if the set A is absolutely convex and absorbing, then p_A is a seminorm on X and

$$\operatorname{aint}(A)=\{x\in X: p_A(x)<1\}\subset A\subset\{x\in X: p_A(x)\le 1\}=\operatorname{acl}(A). \tag{2.12}$$

Proposition 2.25 ([9], P. 2.5.6). *Let A be an absorbing absolutely convex subset of an ordered vector space X.*

1. *If $[A]\subset\gamma A$, then the seminorm p_A is γ-normal.*
2. *If $\forall y\in A,\ [0;y]\subset\gamma A$, then the seminorm p_A is γ-monotone.*
3. *If $\forall y\in A,\ [-y;y]\subset\gamma A$, then the seminorm p_A is γ-absolutely monotone.*

Based on Theorem 2.21 and Proposition 2.25 one can give further characterizations of normal cones in LCS.

Theorem 2.26 ([9], [22] and [25]). *Let (X,τ) be a LCS ordered by a cone K. T.f.a.e.*

1. *The cone K is normal.*
2. *There exists $\gamma>0$ and a family P of γ-normal seminorms generating the topology τ of X.*
3. *There exists $\gamma>0$ and a family P of γ-monotone seminorms generating the topology τ of X.*
4. *There exists $\gamma>0$ and a family P of γ-absolutely monotone seminorms generating the topology τ of X.*

All the above equivalences hold also with $\gamma=1$ in all places.

The following theorem shows the implications the normality has on an arbitrary family of seminorms generating the topology.

Theorem 2.27 ([9]). *Let (X,P) be a LCS ordered by a cone K, where P is a family of seminorms generating the topology of X. T.f.a.e.*

1. *The cone K is normal.*
2. *For each $p\in P$ there exist a finite subset F_p of P and a number $\gamma>0$ such that*
$$\forall x,y,z\in X,\quad x\le z\le y \implies p(z)\le\gamma\max\left\{\max\{q(x),q(y)\}: q\in F_p\right\}.$$
3. *For each $p\in P$ there exist a finite subset F_p of P and a number $\gamma>0$ such that*
$$\forall x,y\in X,\quad 0\le x\le y \implies p(x)\le\gamma\max\left\{q(y): q\in F_p\right\}.$$
4. *For each $p\in P$ there exist a finite subset F_p of P and a number $\gamma>0$ such that*
$$\forall x,y\in X,\quad -y\le x\le y \implies p(x)\le\gamma\max\left\{q(y): q\in F_p\right\}.$$

2.4. **Normal cones in normed spaces.** We shall consider now characterizations of normality in the case of normed spaces. For a normed space $(X, \|\cdot\|)$, let $B_X = \{x \in X : \|x\| \le 1\}$ be its closed unit ball and $S_X = \{x \in X : \|x\| = 1\}$ its unit sphere.

Theorem 2.28 ([13] and [17]). *Let K be a cone in a normed space $(X, \|\cdot\|)$. T.f.a.e.*

1. *The cone K is normal.*
2. *There exists a monotone norm $\|\cdot\|_1$ on X equivalent to the original norm $\|\cdot\|$.*
3. *For all sequences (x_n), (y_n), (z_n) in X such that $x_n \le z_n \le y_n$, $n \in \mathbb{N}$, the conditions $\lim_n x_n = x = \lim_n y_n$ imply $\lim_n z_n = x$.*
4. *The order convex hull $[B_X]$ of the unit ball is bounded.*
5. *The order interval $[x; y]$ is bounded for every $x, y \in X$.*
6. *There exists $\delta > 0$ such that $\forall x, y \in K \cap S_X,\ \|x + y\| \ge \delta$.*
7. *There exists $\gamma > 0$ such that $\forall x, y \in K,\ \|x + y\| \ge \gamma \max\{\|x\|, \|y\|\}$.*
8. *There exists $\lambda > 0$ such that $\|x\| \le \lambda\|y\|$, for all $x, y \in K$ with $x \le y$, (a norm satisfying this condition is called semi-monotone,* [13]*).*

We notice also the following result, which can be obtained as a consequence of a result of T. Andô on ordered locally convex spaces (see [3, Theorem 2.10]).

Proposition 2.29 ([3], C. 2.12). *Let X be a Banach space ordered by a closed generating cone X_+ and B_X its closed unit ball. Then*

$$B_X \cap X_+ - B_X \cap X_+$$

is a neighborhood of 0.

The following notions are inspired by Cantor's theorem on the convergence of bounded monotone sequences of real numbers.

Let X be a Banach space ordered by a cone K. The cone K is called

- *regular* if every increasing and order-bounded sequence in X is convergent;
- *fully regular* if every increasing and norm-bounded sequence in X is convergent.

By Proposition 2.17 if X is a regular normed lattice, then every countable subset of X has a supremum.

These notions are related in the following way.

Theorem 2.30 ([17], Theorems 2.2.1 and 2.2.3). *If X is a Banach space ordered by a cone K, then*

$$K \text{ fully regular} \implies K \text{ regular} \implies K \text{ normal.}$$

If the Banach space X is reflexive, then the reverse implications hold too, i.e. both implications become equivalences.

2.5. **Dual pairs.** Consider a dual pair $\langle X, Y\rangle$ of vector spaces. This means that X, Y are vector spaces for which is defined a bilinear mapping $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{R}$ such that

$$\begin{aligned} &\text{(i)} \quad \forall y \in Y,\ \langle x, y\rangle = 0 \implies x = 0, \text{ and} \\ &\text{(ii)} \quad \forall x \in X,\ \langle x, y\rangle = 0 \implies y = 0, \end{aligned}$$

A typical example of dual pair is formed by a Hausdorff LCS (X, τ) and its dual space $Y = X^*$, with the duality mapping $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ defined by

$$\langle x, f\rangle = f(x) \quad \text{for} \quad (x, f) \in X \times X^*.$$

For each fixed $y \in Y$, $\langle\cdot, y\rangle : X \to \mathbb{R}$ is a linear functional on X and for each $x \in X$, $\langle x, \cdot\rangle : Y \to \mathbb{R}$ is a linear functional on Y. A topology τ on X is called *compatible with the duality* $\langle X, Y\rangle$ if for each continuous linear functional $f : (X, \tau) \to \mathbb{R}$ there exists $y \in Y$ s.t. $f(x) = \langle x, y\rangle$, $x \in X$. In this case one says simply that the dual of (X, τ) is Y, $(X, \tau)^* = Y$. A similar definition can be given for Y. The weakest of these topologies is $\sigma(X, Y)$, the locally convex topology on X generated by the family $\{p_y : y \in Y\}$ of seminorms, where for $y \in Y$, $p_y : X \to \mathbb{R}$ is defined by $p_y(x) = |\langle x, y\rangle|$, $x \in X$. The strongest of these topologies is the so called *Mackey topology*, denoted by $\mu(X, Y)$, which is the topology of uniform convergence on each $\sigma(Y, X)$-compact subset of Y. Denote by $\mathcal{K}_Y$ the family of all $\sigma(Y, X)$-compact subsets of Y. Then the Mackey topology $\mu(X, Y)$ is generated by the family $\{p_K : K \in \mathcal{K}_Y\}$ of seminorms, where for $K \in \mathcal{K}_Y$ the seminorm $p_K : X \to \mathbb{R}$ is defined by $p_K(x) = \sup_{y \in K} |\langle x, y\rangle|$, $x \in X$. Any locally convex topology τ on X compatible with the duality $\langle X, Y\rangle$ satisfies $\sigma(X, Y) \le \tau \le \mu(X, Y)$. Every topology compatible with the duality is the topology of uniform convergence on the sets belonging to a family of $\sigma(Y, X)$-compact subsets of Y. Notice that the closed convex sets in X are the same for all the topologies compatible with the duality, as well as the bounded sets. There is another topology on X, called the *strong topology*, denoted by $\beta(X, Y)$, which is the topology of uniform convergence on each $\sigma(Y, X)$-bounded subset of Y. A subset M of Y is called $\sigma(Y, X)$-bounded if $\sup_{y \in M} |\langle x, y\rangle| < \infty$ for every $x \in X$. Denote by $\mathcal{M}_Y$ the family of all $\sigma(Y, X)$-bounded subsets of Y. Then the strong topology $\beta(X, Y)$ is generated by the family $\{p_M : M \in \mathcal{M}_Y\}$ of seminorms, where for $M \in \mathcal{M}_Y$ the seminorm $p_M : X \to \mathbb{R}$ is defined by $p_M(x) = \sup_{y \in M} |\langle x, y\rangle|$, $x \in X$. This topology is not always compatible with the duality $\langle X, Y\rangle$. In the case of a dual pair $\langle X, X^*\rangle$ formed by a normed space X and its dual X^*, the strong topology on the dual space X^* is the norm topology. Consequently, it is compatible with the duality $\langle X, X^*\rangle$ only for reflexive Banach spaces. By analogy, a LCS X for which $(X^*, \beta(X^*, X))^* = X$ is called semi-reflexive.

See [20], or [25], for details.

If K is a wedge in X, then the dual wedge K' is defined by

$$K' = \{y \in Y : \langle x, y\rangle \ge 0 \text{ for all } x \in K\}.$$

Proposition 2.31 ([3] and [22]). *Let $\langle X, Y\rangle$ be a dual pair and K a wedge in X.*

1. *The wedge K is closed in X for a topology τ compatible with the duality X, Y iff*
$$x \in K \iff \forall y \in K', \ \ \langle x, y\rangle \ge 0.$$
2. *A point x is not in the $\sigma(X, Y)$-closure of K iff there is $y \in K'$ s.t. $\langle x, y\rangle < 0$.*
3. *If K is a cone in X, then the closure $\overline{K}$ of K w.r.t. a topology τ compatible with the duality $\langle X, Y\rangle$ is a cone iff K' is $\sigma(Y, X)$-dense in Y.*
4. *K is a $\sigma(X, Y)$-normal cone iff the dual cone K' is generating in Y, that is $K' - K' = Y$.*
5. *The dual wedge K' is always $\sigma(Y, X)$-closed.*
6. *The dual wedge K' is a cone iff $K - K$ is $\sigma(X, Y)$-dense in X.*
7. *If the wedge K is $\sigma(X, Y)$-closed, then $K' - K'$ is $\sigma(Y, X)$-dense in Y.*

2.6. **Bases for cones.** A nonempty convex subset B of a cone K is called a base for K if for every $x \in K \setminus \{0\}$ there exists a unique pair $(t, b) \in (0; \infty) \times B$ such that $x = tb$.

The following proposition contains some properties of cones with basis.

Proposition 2.32 ([22]). *Let X be a vector space ordered by a cone K admitting a basis B.*

(i) *If $\sum_{i=1}^n \alpha_i b_i = 0$ for some $b_i \in B$ and $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$, then $\sum_{i=1}^n \alpha_i = 0$;*
(ii) *If for some $b, b' \in B$ and $\alpha, \alpha' \in \mathbb{R}$, $\alpha b \le \alpha b'$ ($\alpha b < \alpha b'$), then $\alpha \le \alpha'$ (resp. $\alpha < \alpha'$);*
(iii) *If for some $x_0, y_0 \in K$, $nx_0 \le y_0$ for all $n \in \mathbb{N}$, then $x_0 = 0$. If further X is a vector lattice, then the order is Archimedean.*

The following characterization gives a method to obtain bases.

Theorem 2.33 (see [3] or [9])**.** *Let K be a cone in a vector space X. A subset B of K is a base for B iff there exists a strictly positive linear functional $f : X \to \mathbb{R}$ such that*

$$B = \{x \in K : f(x) = 1\}. \tag{2.13}$$

Moreover, if $f : X \to \mathbb{R}$ is a strictly positive linear functional, then for every $\alpha > 0$ the set

$$B = \{x \in K : f(x) = \alpha\}$$

is a base for K.

The proof goes the following way. If B is a base, then for every $y \in K,\ y \neq 0,$ there exists a unique pair $(g(y), b) \in (0; \infty) \times B$ s.t. $y = g(y)\, b$. Putting $g(0) = 0$ it follows that the so defined functional $g : K \to [0; \infty)$ is additive, so that, by Proposition 3.4, it admits a linear extension $f : X \to \mathbb{R}$ which is strictly positive. It is obvious that f satisfies (2.13). The proof of the converse proceeds by direct verification.

Remark 2.34. It is obvious that if B is a base for a cone K, then $0 \notin K$. In fact a stronger condition holds: $0 \notin \mathrm{aff}(B)$, where $\mathrm{aff}(B)$ stands for the affine hull of B.

In fact, a subset B of a vector space X is a base for the cone $K = \mathrm{cone}(B)$ it generates iff the following conditions hold:

(i) B is nonempty and convex,

(ii) $0 \notin \mathrm{aff}(B)$.

Here

$$\mathrm{cone}(B) = \{tB : t \geq 0,\, b \in B\}.$$

Indeed, by the above theorem there exists a strictly positive linear functional f on X such that (2.13) holds. It follows that $\mathrm{aff}(B)$ is contained in the hyperplane $\{x \in X : f(x) = 1\}$, which, obviously, does not contain 0.

As it is expected, in the case of ordered LCS the characterizing functional is further continuous.

Theorem 2.35 ([9], Th. 2.5.5, and [19], Th. 3.8.4)**.** *Let X be an ordered LCS. T.f.a.e.*

1. *The positive cone X_+ has a base B with $0 \notin \overline{B}$.*
2. *There exists a strictly positive continuous linear functional f on X such that*
$$B = \{x \in K : f(x) = 1\}.$$
3. *The dual cone X_+^* has nonempty interior w.r.t. the strong topology $\beta(X^*, X)$.*

Proposition 2.36 ([9] Th. 2.2.4 and 2.4.3)**.** *If a wedge K in a Hausdorff TVS is generated by a closed bounded base B, then K is a closed normal cone.*

3. Linear operators on ordered vector spaces

3.1. **Classes of linear operators.** Let (X, K), (Y, C) be vector spaces ordered by the cones K, C. A linear operator $T : X \to Y$ is called

- *positive* (with the notation $T \geq 0$) if $T(x) \leq_C 0_Y$ whenever $x \geq_K 0_X$;
- *strictly positive* (with the notation $T > 0$) if $T(x) >_C 0_Y$ whenever $x >_K 0_X$;
- *regular* if $T = T_1 - T_2$ with T_1, T_2 positive operators;
- *order bounded* if T maps order bounded sets in X into order bounded sets in Y.

We shall use the notation:

$\mathcal{L}(X, Y)$ for the space of all linear operators from X to Y;
$\mathcal{L}_r(X, Y)$ for the space of all regular linear operators from X to Y;
$\mathcal{L}_b(X, Y)$ for the space of all order bounded linear operators from X to Y;

$\mathcal{L}_+(X,Y)$ for the wedge of all positive linear operators from X to Y.

Note. In the following the term "operator" will mean always a linear operator.

The following (possibly strict) inclusions hold true

$$\mathcal{L}_r(X,Y) \subset \mathcal{L}_b(X,Y) \subset \mathcal{L}(X,Y).$$

Proposition 3.1 ([3] and [22]). *Let $(X,K),(Y,C)$ be ordered vector spaces.*

1. *If the cone K is generating, then $\mathcal{L}_+(X,Y)$ is a cone.*
2. *If $(X,K),(Y,C)$ are vector lattices, with Y order complete, then $\mathcal{L}_b(X,Y)$ is an order complete vector lattice with respect to the cone $\mathcal{L}_+(X,Y)$. Since $\mathcal{L}_b(X,Y)$ is a vector lattice it follows $\mathcal{L}_b(X,Y)=\mathcal{L}_r(X,Y)$.*

One says that an ordered vector space X has the *decomposition property* (or *Riesz decomposition property*) if, for all $x,y\geq 0$,

$$0\leq z\leq x+y \implies \exists z_1,z_2,\ 0\leq z_1\leq x,\ 0\leq z_2\leq y,\ z=z_1+z_2, \tag{3.1}$$

a condition equivalent to

$$[0;x]+[0;y]=[0;x+y]. \tag{3.2}$$

Any vector lattice has the decomposition property, but there are ordered vector spaces with the decomposition property which are not vector lattices (see [22]).

Using Riesz decomposition property, which is strictly weaker than that of being a vector lattice, one can give a slight extension to Proposition 3.1.2.

Theorem 3.2 (Riesz-Kantorovich, see [3]). *If X be an ordered vector space with the decomposition property and the cone X_+ generating, Y an order complete vector lattice. Then $\mathcal{L}_b(X,Y)$ is an order complete vector lattice with respect to the cone $\mathcal{L}_+(X,Y)$, so that $\mathcal{L}_b(X,Y)=\mathcal{L}_r(X,Y)$ (because $\mathcal{L}_b(X,Y)$ is a vector lattice).*

The lattice operations in $\mathcal{L}_b(X,Y)$ are given by the formulae:

$$\begin{aligned}
&(S\vee T)(x)=\sup\{S(y)+T(z): x=y+z,\ y,z\in X_+\},\\
&(S\wedge T)(x)=\inf\{S(y)+T(z): x=y+z,\ y,z\in X_+\},\\
&|S|(x)=\sup\{S(y): -y\leq x\leq y\}.
\end{aligned}$$

Riesz decomposition property is related to other important property attributed to Riesz too. An ordered vector space X is said to have the *Riesz interpolation property* (or simply, the *interpolation property*) if for any two finite subsets A,B of X such that $A\leq B$ there exists $z\in Z$ satisfying $A\leq z\leq B$.

Theorem 3.3 (F. Riesz, see [3]). *If (X,K) is an ordered vector space, then the following assertions are equivalent.*

1. *The space X has the interpolation property.*
2. *For any two element subsets A,B of X such that $A\leq B$ there exists $z\in Z$ satisfying $A\leq z\leq B$.*
3. *The space X has the decomposition property.*
4. *If $\{x_1,x_2,y_1,y_2\}\subset K$ are such that $x_1+x_2=y_1+y_2$, then there exist the vectors $\{z_{i,j}: i,j=1,2\}\subset K$ satisfying*

$$x_i=z_{i1}+z_{i2},\ i=1,2, \quad \textit{and} \quad y_j=z_{1j}+z_{2j},\ j=1,2.$$

5. *If* $\{x_i : i = 1, \dots, m\}$, $\{y_j : j = 1, \dots, n\}$ *and are two sets of positive elements such that* $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j$, *then there exist the vectors* $\{z_{i,j} : i = 1, \dots, m,\ j = 1, \dots, n\} \subset K$ *satisfying*
$$x_i = \sum_{j=1}^n z_{ij},\ \ i = 1, \dots, m, \quad \text{and} \quad y_j = \sum_{i=1}^m z_{ij},\ \ j = 1, \dots, n.$$

3.2. **Extensions of positive operators.** Various extension results for positive operators play a key role in the theory of these operators and in applications.

Proposition 3.4 ([3])**.** *Let* (X, K), (Y, C) *be vector spaces ordered by the wedges* K, C, *with* C *Archimedean, and* $S : K \to C$ *an additive operator. Then there exists a positive linear operator* $T : X \to Y$ *extending* S. *The extension is unique to the subspace* $K - K$ *generated by the wedge* K.

Consequently, if the wedge K *is generating then the extension to* X *is unique.*

A mapping $f : (X, K) \to (Y, C)$ is called

- *convex* if

$$\forall x, y \in X,\ \forall t \in [0; 1] \quad f((1 - t)x + ty) \le (1 - t)f(x) + tf(y). \tag{3.3}$$

- *sublinear* if

$$\begin{aligned} &\text{(i)} \quad f(x + y) \le f(x) + f(y),\\ &\text{(ii)} \quad f(tx) = tf(x), \end{aligned} \tag{3.4}$$

for all $x, y \in X$ and all $t \ge 0$, that is it is (i) subadditive and (ii) positively homogeneous.

Obviously, a sublinear map is convex.

One says that a mapping $f : A \to Y$ is dominated by $g : X \to Y$ on $A \subset X$ if
$$\forall x \in A, \quad f(x) \le g(x).$$

The following result is the analog of Hahn-Banach extension theorem. Its proof is the same as the proof of the classical Hahn-Banach theorem, only the context is different.

Theorem 3.5 (Hahn-Banach Extension Theorem, [3])**.** *Let* X *be a real vector space,* Y *an order complete vector lattice,* $p : X \to Y$ *a convex mapping and* Z *a linear subspace of* X.

If $S : Z \to Y$ *is a linear operator dominated by* p *on* Z, *then there exists a linear operator* $T : X \to Y$ *dominated by* p *on* X *such that* $T|_Z = S$.

This theorem has as consequence the following useful extension results for positive operators.

Corollary 3.6. *Let* (X, K), (Y, C) *be vector lattices, with* Y *order complete,* $T : X \to Y$ *a positive operator and* Z *a vector sublattice of* X. *Then for any linear operator* $S : Z \to Y$ *such that* $0 \le S(z) \le T(z)$ *for all* $z \in K \cap Z$ *there exists a linear operator* $\tilde{S} : X \to Y$ *such that* $0 \le \tilde{S}(x) \le T(x)$ *for all* $x \in K$.

The following extension theorem was proved by L. V. Kantorovich in 1930. A subspace Z of an ordered vector space (X, K) is called *majorizing* (or *cofinal*) if for every $u \in K$ there exists $z \in Z$ with $u \le z$, or, equivalently, if
$$K \subset Z - K.$$

It is easy to prove that

Lemma 3.7 ([19])**.** *Let* (X, K) *be an ordered vector space.*

1. *If* Z *is a majorizing subspace of* X, *then*
$$Z + K = Z - K = Z + K - K.$$

2. *If the cone K is generating and Z is a majorizing subspace of X, then $X = Z - K$, that is for every $x \in X$ there exists $z \in Z$, $z \geq x$.*

Theorem 3.8 (Kantorovich's Extension Theorem). *Let (X, K), (Y, C) be ordered vector spaces, with Y an order complete vector lattice, and Z a majorizing subspace of X. Then for every positive operator $S : Z \to Y$ there exists a positive operator $T : X \to Y$ extending S.*

The idea of the proof is to apply Theorem 3.5 to S and to the functional

$$p(x) = \inf\{S(z) : z \in Z,\ x \leq z\}, \quad (x \in Z + K - K),$$

which is sublinear and dominates S on Z (in fact $p(z) = S(z)$ for $z \in Z$). One obtains a positive linear extension $\tilde{S}$ to the subspace $W = Z + K - K$. Writing $X = W \dot{+} W_1$ (direct algebraic sum with a subspace W_1 of X) and putting $T(w + w_1) = \tilde{S}(w)$, one obtains a positive linear extension to X.

The following theorem extends a well known result concerning linear functionals.

Proposition 3.9 ([3]). *Let (X, K) be a vector lattice, (Y, C) an order σ-complete vector lattice and $T : X \to Y$ a positive operator. Then for every $x_0 \in K$ there exists a positive operator $S : X \to Y$ such that*

$$\begin{aligned} &\text{(i)} \quad \forall x \in K,\ 0 \leq S(x) \leq T(x); \\ &\text{(ii)} \quad S(x_0) = T(x_0); \\ &\text{(iii)} \quad \forall x \in X,\ |x| \wedge x_0 = 0 \ \Rightarrow\ S(x) = 0. \end{aligned} \tag{3.5}$$

3.3. **The case of linear functionals.** In the case of linear functionals there are some specific results. Recall that for a linear functional $f : X \to \mathbb{R}$ the kernel of f, defined by

$$\ker f = \{x \in X : f(x) = 0\}, \tag{3.6}$$

is a maximal linear subspace of X.

Proposition 3.10. *Let (X, K) be an ordered vector space and $f : X \to \mathbb{R}$ a linear functional. Then*

1. *If f is positive, then $\ker f$ is order convex.*
2. *If $\ker f$ is order convex, then either f or $-f$ is positive.*

As a consequence of Theorem 3.2 one obtains.

Proposition 3.11 (F. Riesz, see [22]). *If (X, K) is an ordered vector space with the decomposition property and the cone K generating, then the space $\mathcal{L}_b(X, \mathbb{R})$ is an order complete vector lattice with respect to the order determined by the cone $\mathcal{L}_+(X, \mathbb{R})$.*

We mention the following extension results for linear functionals.

Theorem 3.12 (H. Bauer, I. Namioka, see [3] and [9]). *Let (X, K) be an ordered vector space, Z vector subspace of X and $f : Z \to \mathbb{R}$ a linear functional.*

Then there exists a positive linear functional $F : X \to \mathbb{R}$ extending f iff there exists a convex absorbing subset B of X such that

$$\forall x \in Z \cap (B - K), \quad f(x) \leq 1. \tag{3.7}$$

Corollary 3.13 (H. Bauer, F. F. Bonsall, I. Namioka, see [19]). *Let (X, K) be an ordered vector space. If Z is a majorizing vector subspace of X, then every positive linear functional $f : Z \to \mathbb{R}$ admits a positive linear extension $F : X \to \mathbb{R}$.*

The following corollary expresses the possibility of extending positive linear functionals from subspaces containing an order unit.

Corollary 3.14 (Krein-Rutman, see [19] and [9]). *Let (X, K) be an ordered vector space.*

1. *If Z is a vector subspace of X containing an order unit of X, then every positive linear functional $f : Z \to \mathbb{R}$ admits a positive linear extension $F : X \to \mathbb{R}$.*
2. *If the space X contains an order unit, then there exists a nonzero positive linear functional on X.*

3.4. **Order units and the continuity of linear functionals.** If X, Y are TVS, one denotes by $L(X, Y)$ the space of all continuous linear operators from X to Y and by $X^* = L(X, \mathbb{R})$ the algebraic topological dual of X, i.e. the space of all continuous linear functionals on X.

For a TVS (X, τ) ordered by a cone $K = X_+$ the following notations will be used

- $X^\# = \mathcal{L}(X, \mathbb{R})$ - the algebraic dual of X;
- $X_+^\# = \mathcal{L}_+(X, \mathbb{R}) =$ the set of all positive linear functionals - the algebraic dual cone to X_+;
- $X_b^\# = \mathcal{L}_b(X, \mathbb{R}) =$ the space of all order bounded linear functionals;
- $X_r^\# = \mathcal{L}_r(X, \mathbb{R}) =$ the space of all regular linear functionals;
- $X_+^* = X_+^\# \cap X^* =$ the set of all positive continuous linear functionals - the topological dual cone to X_+;

We present now some results on order units in ordered TVS.

Proposition 3.15 ([3])**.** *Let (X, τ) be an ordered TVS.*

1. *u is an interior point of the cone X_+ iff $[-u; u]$ is a neighborhood of 0.*
2. *If u is an interior point of the cone X_+, then*
   (i) *u is an order unit, and*
   (i) *$X_b^\# \subset X^*$, that is every order bounded linear functional is continuous.*
3. *Suppose that the cone X_+ has a base $B = \{x \in X_+ : f(x) = 1\}$ determined by a strictly positive continuous linear functional f and let $H = \{x \in X : f(x) = 1\}$ be the closed hyperplane determined by f containing B.*

   Then $u \in B$ is an interior point of the cone X_+ iff it is an interior point of B relative to H.
4. *If there exists a strictly positive continuous linear functional f on X then the set of all strictly positive continuous linear functionals is dense in X_+^* for any vector topology on X^*.*

In the case of a complete metrizable TVS the following equivalences hold.

Theorem 3.16 ([3])**.** *Let (X, τ) be an ordered complete metrizable TVS. Then for $u > 0$ the following assertions are equivalent.*

1. *u is an order unit.*
2. *u is an algebraic interior point of X_+.*
3. *u is an interior point of X_+.*

3.5. **Locally order bounded TVS.** In this subsection we present following [9] some results on locally order bounded TVS. An ordered TVS is called locally order bounded if it possesses an order bounded 0-neighborhood.

Proposition 3.17. *Let X be an ordered TVS. Then*

1. *The space X is locally order bounded iff the positive cone X_+ is solid (i.e. $\operatorname{int}(X_+) \neq \emptyset$).*
2. *If the space X is locally order bounded, then every order unit belongs to $\operatorname{int}(X_+)$.*
3. *If the space X is locally order bounded, then every order bounded linear functional is continuous.*
4. *Suppose that the space X is locally order bounded and Z is a subspace of X s.t. $Z \cap \operatorname{int}(X_+) \neq \emptyset$. Then every linear functional $f : Z \to \mathbb{R}$ satisfying*
$$f(z) \geq 0 \quad \text{for all} \quad z \in Z \cap \operatorname{int}(X_+),$$
   has a positive continuous linear extension to X.

4. Extremal structure of convex sets and elements of Choquet theory

4.1. **Faces and extremal vectors.** Let X be a vector space and C a nonempty convex subset of X. A convex subset Z of C is called a *face* (or an *extremal subset*) of C if $(1-t)x+ty \in Z$ for $x, y \in C$ and some $t \in (0;1)$ implies $x, y \in Z$, and so, by the convexity of Z, $[x;y]_a \subset Z$. Recall that we denote by $[x;y]_a$ the algebraic segment determined by x, y, $[x;y]_a = \{(1-t)x+ty : t \in [0;1]\}$. If $Z = \{z\}$ is a one point face of C, then z is called an *extreme point* of C. A face of the form $x_0 + \mathbb{R}_+ z_0$ for some $x_0, z_0 \in C$, with $z_0 \neq 0$, is called an *extreme ray* of C. If C is a cone, then the extreme rays are of the form $\mathbb{R}_+ z_0$ for some $z_0 \in C$ with $z_0 \neq 0$. Denote by $\operatorname{ext}(C)$ the set of extreme points of the set C.

The following proposition contains some simple properties of faces.

Proposition 4.1. *Let X be a vector space and C a nonempty convex subset of X.*

1. *If Z is a face of C and W is a face of Z, then W is a face of C.*
2. *An extreme point of a face of C is an extreme point of C.*
3. *A nonempty intersection of nonempty faces of C is a face of C.*
4. *If f is a nonzero linear functional attaining its maximum (or minimum) m on C, then the set $\{x \in C : f(x) = m\}$ is a face of C. In other words, the intersection of C with a support hyperplane is a face of C.*
5. *If C is a cone, then 0 is the only extreme point of C.*
6. *Every face of a cone C is a subcone of C.*

Extreme rays of cones can be characterized in the following way.

Proposition 4.2 ([11], P. 25.6 and P. 25.7)**.** *Let W be a wedge in a vector space X and $\Delta \subset W$ a ray. T.f.a.e.*

1. *Δ is an extreme ray.*
2. *If $(x;y)_a$ is an open algebraic interval in X intersecting Δ in a non-zero point, then $(x;y)_a \subset \Delta$.*
3. *$W \setminus \Delta$ is convex.*

If W is further a cone, then the above conditions are also equivalent to

4. *If $x \in W$, $y \in \Delta$ are such that $x \leq y$, then $x \in \Delta$.*

Example 4.3 ([11], Exercise 25.5)**.** If T is a Hausdorff locally compact space without isolated points and containing at least 2 points, then the positive cone in $C(T)$ has no extreme rays.

This happens, for instance, for $T = [a;b]$, or $T = [a;\infty)$ (intervals in $\mathbb{R}$).

4.2. **Extreme points, extreme rays and Krein-Milman's Theorem.** A famous result concerning the extreme points is Krein-Milman theorem.

Theorem 4.4 (Krein-Milman, see[20])**.** *Let X be a locally convex space and C a nonempty compact convex subset of X.*

1. *C coincides with the closed convex hull of its extreme points.*
2. *If Z is a subset of C such that $\overline{\operatorname{co}}(Z) = C$, then $\operatorname{ext}(C) \subset \overline{Z}$.*

Remark 4.5. The second assertion of the above theorem was proved by D. Milman.

We mention the following result of Bauer concerning convex functions.

Theorem 4.6 (Bauer Maximum Principle, [11], Th. 25.9)**.** *Let Y be a compact convex subset of a Hausdorff LCS. Then for every convex usc function $f : Y \to \mathbb{R}$ there exists an extreme point of Y (not necessarily unique) at which f takes its maximum value.*

The idea of the proof is simple. Since f is usc and Y is compact, f attains its maximum value m at some point $x_0 \in Y$. There exists a maximal convex subset C of Y containing x_0 and such that

$f(y) = m$ for every $y \in C$. It follows that C is a closed face of Y and so it has an extreme point which is an extreme point of Y too.

The extreme points of a base B and the extreme rays are related the following way.

Theorem 4.7 ([3])**.** *Let B be a base for a cone K in a vector space X. Then b is an extreme point of B iff $\mathbb{R}_+b$ is an extreme ray of K.*

In order to formulate a Krein-Milman type theorem for cones we need the following result.

Theorem 4.8 ([20])**.** *A cone in a Hausdorff locally convex space X is locally compact iff it is the cone generated by a compact convex set B which does not contain 0. Furthermore, B can be chosen to be a subset of a closed hyperplane.*

Such a cone is always closed.

Based on these two theorems one can prove a Krein-Milman type theorem for cones.

Theorem 4.9 (Krein-Milman, see [20])**.** *A locally compact cone in a Hausdorff locally convex space agrees with the closed convex hull of its extreme rays.*

The theorem can be extended to a wider class of locally compact convex set.

Theorem 4.10 (Krein-Milman, see [20])**.** *Every locally compact convex subset of a Hausdorff locally convex space which does not contain lines agrees with the closed convex hull of its extreme points and extreme rays.*

4.3. **Regular Borel measures and Riesz' Representation Theorem.** A *Borel subset* of a locally compact Hausdorff topological space T is an element of the σ-algebra $\mathcal{B}(T)$ generated by the family of open subsets of T. A *Baire subset* of a locally compact Hausdorff topological space T is an element of the σ-algebra $\mathcal{B}_0(T)$ generated by the family of all compact G_δ (the intersection of a countable family of open sets) subsets of T. This is the smallest σ-algebra $\mathcal{B}_0$ of subsets of T making all real-valued continuous functions with compact support $\mathcal{B}_0$-measurable. If T is locally compact and σ-compact, then this is the smallest σ-algebra of subsets of T making all real-valued continuous functions on T $\mathcal{B}_0$-measurable. Since a compact G_δ-set belongs to $\mathcal{B}(T)$, it follows that $\mathcal{B}_0(T) \subset \mathcal{B}(T)$. If T is locally compact and metrizable, then $\mathcal{B}_0(T) = \mathcal{B}(T)$.

Let $\mathcal{A}$ be a σ-algebra of subsets of a set T. A *measure* is a σ-additive map $\mu : \mathcal{A} \to [0; \infty]$ which is not identically ∞ (or, equivalently, such that $\mu(\emptyset) = 0$). A *probability measure* is a measure such that $\mu(T) = 1$. An example of probability measure is the so called Dirac measure, where, for fixed $t \in T$, $\varepsilon_t : \mathcal{A} \to [0; \infty)$ is defined by $\varepsilon_t(A) = 1$ if $t \in A$ and $\varepsilon_t(A) = 0$ if $t \notin A$, for $A \in \mathcal{A}$.

A signed measure is a countably additive mappings $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ with $\mu(\emptyset) = 0$ and which takes at most one of the values $-\infty, \infty$. The total variation of μ is defined by

$$|\mu|(A) = \sup \sum_{i=1}^{n} |\mu(A_i)|, \tag{4.1}$$

where the supremum is taken over all finite measurable decompositions of A, $A = \cup_{i=1}^{n} A_i$, $A_i \in \mathcal{A}$, $A_i \cup A_j = \emptyset$ for $i \neq j$, $n \in \mathbb{N}$. The measure μ is called with *finite variation* if

$$\|\mu\| := |\mu|(T) < \infty. \tag{4.2}$$

It follows that $\mu^+ := \frac{1}{2}(|\mu| + \mu)$ and $\mu^- := \frac{1}{2}(|\mu| - \mu)$ are (positive) measures on $\mathcal{A}$ and

$$\mu = \mu^+ - \mu^- \quad \text{and} \quad |\mu| = \mu^+ + \mu^-. \tag{4.3}$$

Remark 4.11. The decomposition (4.3) agrees with the Jordan decomposition of a signed measure, which can be obtained via Hahn decomposition of a signed measure space $(T, \dashv, \mu)$ (see, e.g., [15]).

One says that a measure $\mu : \mathcal{A} \to [0;\infty]$ is *supported* by a set $S \in \mathcal{A}$ if $\mu(T \setminus S) = 0$. The *support* $S(\mu)$ of a Borel measure μ is the complement to the largest open subset G of T for which $\mu(G) = 0$. It follows that the support is a closed set and a Borel measure μ is supported by S iff $S(\mu) \subset S$. A linear combination $\mu = \sum_{i=1}^{n} \alpha_i \varepsilon_{t_i}$ of Dirac measures is called a *discrete measure*. Obviously, its support is the set $\{t_1, \ldots, t_n\}$.

If T is topological space, then a measure μ defined on $\mathcal{B}(T)$ is called a *Borel measure*. A Borel measure μ is called *regular* if $\mu(K) < \infty$ for every compact subset K of T and

$$\begin{aligned} &\text{(i)} \quad \mu(B) = \inf\{\mu(U) : U \text{ open and } B \subset U\}, \quad \text{for every } B \in \mathcal{B}(T), \ \text{ and} \\ &\text{(ii)} \quad \mu(U) = \sup\{\mu(K) : K \text{ compact and } K \subset U\}, \quad \text{for every } U \subset T \text{ open.} \end{aligned}$$

for every $B \in \mathcal{B}(T)$. It follows

$$\mu(B) = \sup\{\mu(K) : K \text{ compact and } K \subset B\},$$

for every $B \in \mathcal{B}(T)$ with $\mu(B) < \infty$.

A signed measure $\mu : \mathcal{B}(T) \to \overline{\mathbb{R}}$ is called a Borel signed measure. A Borel signed measure is called regular if $|\mu|$ is a regular Borel measure (or, equivalently, if both μ^+ and μ^- are regular Borel measures). Denote by $\mathcal{M}(T)$ the space of all regular Borel signed measures. If T is a compact Hausdorff space, and μ is a regular Borel signed measure on T, then $|\mu|(T) < \infty$ and (4.2) defines a complete norm on $\mathcal{M}(T)$, that is $\mathcal{M}(T)$ is a Banach space w.r.t. the norm (4.2). This will follow from Riesz' representation theorem.

For a compact Hausdorff space T one denotes by $C(T)$ the Banach space of all continuous functions $f : T \to \mathbb{R}$ with the sup-norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in T\}, \quad f \in C(T). \tag{4.4}$$

The support of a function $f \in C(T)$ is the set

$$\operatorname{supp}(f) = \operatorname{cl}(\{t \in T : f(t) \neq 0\}. \tag{4.5}$$

One denotes by $C_{00}(T)$ the vector space of all continuous functions with compact support.

Let T be locally compact Hausdorff. One says that a function $f : T \to \mathbb{R}$ vanishes at infinity if

$$\forall \varepsilon > 0,\ \exists K_\varepsilon \subset T,\ K_\varepsilon \text{ compact}, \quad \textit{s.t.} \quad \forall t \in T \setminus K_\varepsilon,\ \ |f(t)| < \varepsilon.$$

The space of all continuous functions on T vanishing at infinity is denoted by $C_0(T)$. It is a Banach space w.r.t. the norm (4.4) and $C_0(T) = C(T)$ if T is compact.

The following theorem is one of the cornerstones of mathematical analysis.

Theorem 4.12 (Riesz' Representation Theorem 1)**.** *Let T be a locally compact Hausdorff space. Then for every positive linear functional $\ell : C(T) \to \mathbb{R}$ there exists a unique regular Borel measure $\mu : \mathcal{B}(T) \to [0;\infty)$ such that*

$$\begin{aligned} &\text{(i)} \quad \ell(f) = \int_T f d\mu, \quad \forall f \in C(T), \quad \textit{and} \\ &\text{(i)} \quad \|\ell\| = \|\mu\|, \end{aligned}$$

where $\|\mu\| = \mu(T)$.

The previous form of the Riesz' representation theorem leads to the general form of continuous linear functionals on $C(T)$.

Theorem 4.13 (Riesz' Representation Theorem 2)**.** *Let T be a compact Hausdorff space. Then for every continuous linear functional $\ell : C(T) \to \mathbb{R}$ there exists a unique regular Borel signed measure $\mu : \mathcal{B}(T) \to \mathbb{R}$ such that*

$$\text{(i)} \quad \ell(f) = \int_T f d\mu, \quad \forall f \in C(T), \quad \textit{and}$$

$$\text{(i)} \quad \|\ell\| = \|\mu\|,$$

where $\|\mu\| = |\mu|(T)$.

4.4. **Radon measures.** In [10] and [11] the theory of integration is developed à la Bourbaki, see [8]. For a Hausdorff locally compact space T one equips $C_{00}(T)$ with the inductive limit topology τ with respect to the family $C_K(T) = \{f \in C_{00}(T) : \operatorname{supp}(f) \subset K\}$, $K \subset T$, K compact, of subspaces of $C_{00}(T)$. It follows that a sequence (f_n) in $C_{00}(T)$ converges to $f \in C_{00}(T)$ w.r.t. τ iff there exists a compact subset K of T s.t. $\operatorname{supp}(f_n) \subset K$, $n \in \mathbb{N}$, $\operatorname{supp}(f) \subset K$, and

$$f_n|_K \overset{K}{\rightrightarrows} f|K \qquad \text{(uniform convergence)}.$$

A *Radon measure* is a continuous linear functional $I : (C(T), \tau) \to \mathbb{R}$. The family of all Radon measures is denoted by $\mathcal{M}(T)$ and the family of all positive Radon measures is denoted by $\mathcal{M}^+(T)$. The continuity of a linear functional $I : (C(T), \tau) \to \mathbb{R}$ is equivalent to the following property: for every compact $K \subset T$ there exists a number $\beta_K > 0$ s.t.

$$|I(f)| \le \beta_K \|f\|_\infty,$$

for all $f \in C_{00}(T)$ with $\operatorname{supp}(f) \subset K$.

Note that a positive linear functional $I : (C(T), \tau) \to \mathbb{R}$ is automatically continuous. Riesz' Representation Theorem tells us that in the case when T is compact $\mathcal{M}(T)$ can be identified with the set of all regular Borel signed measures, and $\mathcal{M}^+(T)$ with the set of all regular Borel measures.

Ordering $X = C_{00}(T)$ as usual with the pointwise order

$$f \le g \overset{\text{def}}{\iff} \forall t \in T,\ f(t) \le g(t),$$

it follows that $\mathcal{M}^+(T)$ is the dual cone X^*_+. The following order properties hold too.

Theorem 4.14 ([10], Th. 11.2)**.** *Let T be a Hausdorff locally compact space. Then the following assertions are true.*

1. *$C_{00}(T)$ is a vector lattice.*
2. *$\mathcal{M}^+(T) := C_{00}(T)^{\#}_+ = C_{00}(T)^*_+$, that is every positive linear functional is continuous (and positive, of course).*
3. *$\mathcal{M}(T) := C_{00}(T)^{\#}_b$, that is the space of Radon measures agrees with the space of all order bounded linear functionals on $C_{00}(T)$.*
4. *$\mathcal{M}(T)$ is a complete lattice.*
5. *Every $\mu \in \mathcal{M}(T)$ can be written as $\mu = \mu^+ - \mu^-$ with $\mu^+ \wedge \mu^- = 0$.*

(See Subsection 3.4 for the notation).

4.5. **Elements of Choquet theory.** Choquet theory (see [23] and [10, 11, 12]) is a far reaching generalizations of Krein-Milman theorem. It deals with representations of convex functions as integrals w.r.t. Borel measures on the set of extreme points of compact convex sets. Besides its intrinsic theoretical interest, the developed theory has substantial applications to various areas of mathematics - approximation theory, potential theory, mathematical analysis, see [4], [5], [7], [11], [12], [23].

If Y is a compact convex subset of $\mathbb{R}^n$, then every point $x \in Y$ can be written as convex combination of $n+1$ extreme points of Y. That is there exists $\alpha_i \ge 0,\ e_i \in \operatorname{ext}(Y),\ i = 1, \dots, n+1,$ s.t. $\sum_{i=1}^{n+1} \alpha_i = 1$

and $x = \sum_{i=1}^{n+1} \alpha_i e_i$. Considering the Dirac measures $\varepsilon_{e_i},\ i = 1, \ldots, n+1,$ it follows that for every linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$

$$\ell(x) = \ell(\sum_{i=1}^{n+1} \alpha_i e_i) = \sum_{i=1}^{n+1} \alpha_i \ell(e_i) = \int_Y \ell d\mu, \tag{4.6}$$

where μ is the probability measure $\mu = \sum_{i=1}^{n+1} \alpha_i \varepsilon_{e_i}$.

Starting form this remark we say that a regular Borel probability measure μ defined on a compact subset Y of a Hausdorff LCS X *represents* a point $x \in X$ if

$$\ell(x) = \int_Y \ell d\mu, \tag{4.7}$$

for every continuous linear functional $\ell \in X^*$. The unique point x satisfying (4.7) is denoted by $r(\mu)$ and is called also the barycenter of the measure μ. Also $\int_Y f d\mu$ is denoted sometimes by $\mu(f)$. If the probability measure μ is discrete, $\mu = \sum_{i=1}^{n} \alpha_i \varepsilon_{y_i}$, with $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, then $r(\mu)$ is the barycenter of its support, $r(\mu) = \sum_{i=1}^{n} \alpha_i y_i$.

Note. In what follows by a **probability measure** we shall mean always a regular Borel probability measure. The set of probability measures on a compact space T will be denoted by $\mathcal{M}^1(T)$ and the set of (positive) Borel measures by $\mathcal{M}^+(T)$. The Dirac measures are the extreme points of the set $\mathcal{M}^1(T)$,

$$\operatorname{ext}\left(\mathcal{M}^1(T)\right) = \{\varepsilon_t : t \in T\}. \tag{4.8}$$

Proposition 4.15 ([23] and [11]). *Let Z be a compact subset of a Hausdorff LCS X s.t. its closed convex hull $Y = \overline{\mathrm{co}}(Z)$ is compact. Then the following assertions hold true.*

1. *For every probability μ on Z there exists a unique point $x = r(\mu) \in Y$ represented by μ.*
2. *The mapping $\mu \mapsto r(\mu)$ is weakly continuous, meaning that for every net $(\mu_i : i \in I)$ of probability measures on Z with barycenters x_i, and a probability measure μ on Z with barycenter x, $\mu_i(f) \to \mu(f)$, for every $f \in C(Z)$, implies $x_i \to x$ in X.*
3. *For every $x \in Y$ there exists a net $(\mu_i : i \in I)$ of discrete probability measures with $S(\mu_i) \subset \operatorname{ext}(Y)$, $i \in I$, and $r(\mu_i) \to x$.*
4. *For every $x \in Y$ there exists a probability measure $\mu \in \mathcal{M}^1(Y)$ with $S(\mu) \subset \operatorname{cl}(\operatorname{ext}(Y))$ and $r(\mu) = x$.*
5. *For every $\mu \in \mathcal{M}^1(Y)$ there exists a net $(\mu_i : i \in I)$ of discrete probability measures with $r(\mu_i) = r(\mu)$, $i \in I$, which converges vaguely to μ.*

Remark 4.16. In probability theory, the convergence $\mu_i(f) \to \mu(f)$, $f \in C(T)$, of a net (μ_i) of probability measures to the probability measure μ is called *vague convergence* (one says also that the net (μ_i) *converges vaguely* to μ). The corresponding topology is called the *vague topology.*

The following proposition gives a characterization of the points in the closed convex closure of a compact set in terms of representing measures.

Proposition 4.17 ([23] and [11]). *Let Z be a compact subset of a Hausdorff LCS. Then a point $x \in X$ belongs to the closed convex hull $Y = \overline{\mathrm{co}}(Z)$ of Z iff there exists a probability measure μ on Z representing x.*

Based on this proposition one can give a formulation of Krein-Milman theorem, Theorem 4.4, in terms of representing measures.

Corollary 4.18 (Krein-Milman Theorem in terms of representing measures)**.** *Any point of a compact convex subset Y of a Hausdorff LCS X is the barycenter of a probability measure supported by the closure of the set of its extreme points.*

It is clear that ε_x is a representing measure for a point x of a convex set Y. If x is not an extreme point, then there exists other probability measures representing x. The following result of Bauer shows that this is a characteristic property of extreme points.

Theorem 4.19 (Bauer's Theorem, see [23])**.** *Let Y be a compact convex subset of a Hausdorff LCS X and $x \in Y$. Then x is an extreme point of Y iff ε_x is the unique probability measure on Y representing x.*

Remark 4.20. In [23] it is shown that combining Proposition 4.17 and Theorem 4.19, one can obtain a proof of the second assertion in Theorem 4.4.

Indeed, let Y be compact convex and $Y = \overline{\mathrm{co}}(Z)$. If x is an extreme point of Y, then, by Proposition 4.17, there exists a probability measure supported by $\overline{Z}$ representing x. By Theorem 4.19, $\mu = \varepsilon_x$, proving that $x \in \overline{Z}$.

As it was shown by Choquet, in the metrizable case the representation can be given in terms of Borel measures.

Theorem 4.21 (Choquet Representation Theorem 1 - the metrizable case, see [23])**.** *Let Y be a compact convex metrizable subset of a Hausdorff LCS X. Then the set of extreme points is a G_δ subset of Y and for every $x \in Y$ there exists a Borel probability measure μ supported by the set of extreme points of Y which represents x.*

Remark 4.22. 1. An essential ingredient in the proof of Theorem 4.21 is the existence of a strictly convex function on a metrizable compact convex subset of a Hausdorff LCS, a property which is in fact equivalent to the metrizability - the existence of a strictly convex function on a compact convex set implies the metrizability of its topology (see [11]).

2. A slightly more general result can be proved concerning the topological behavior of the set $\mathrm{ext}(Y)$: If the set $\mathrm{cl}(\mathrm{ext}(Y))$ is metrizable, then $\mathrm{ext}(Y)$ is G_δ in $\mathrm{cl}(\mathrm{ext}(Y))$ (see [11], C. 27.10).

In the case of non-metrizable compact convex sets the result does not hold in this form. As it is mentioned in [23] there are two ways to circumvent this difficulty: to modify the notion of "supported", or to admit measures which are not Borel. These two ways are illustrated in the next two theorems.

Theorem 4.23 (Choquet Representation Theorem 2 - the non-metrizable case, see [23])**.** *Let Y be a compact convex subset of a Hausdorff LCS X. Then for every $x \in Y$ there exists a Borel probability measure μ representing x which vanishes on every Baire subset of Y disjoint from the set of extreme points of Y.*

The second version in the non-metrizable case is the following.

Theorem 4.24 (Choquet Representation Theorem 3 - the non-metrizable case, see [23])**.** *Let Y be a compact convex subset of a Hausdorff LCS X and let $\mathcal{A}$ be the σ-algebra generated by the Baire subsets of X and by the set $\mathrm{ext}(Y)$. Then for every $x \in Y$ there exists a probability measure μ on $\mathcal{A}$ such that $\mu(\mathrm{ext}(Y)) = 1$.*

4.6. **Maximal measures.** For a compact convex set Y of a Hausdorff LCS X one denotes by $\mathrm{Co}(Y)$ the set of continuous convex real-valued functions defined on Y and by $\mathrm{Af}(Y)$ continuous affine real-valued functions defined on Y. It follows that $-\mathrm{Co}(Y)$ is the set of continuous concave real-valued functions defined on Y. To a bounded function $f : Y \to \mathbb{R}$ one associates two functions

$$\hat{f}(x) = \inf\{g(x) : -g \in \mathrm{Co}(Y),\ f \leq g\}, \tag{4.9}$$

and

$$\check{f}(x) = \sup\{g(x) : g \in \mathrm{Co}(Y),\, g \le f\}, \tag{4.10}$$

the upper, respectively the lower, envelope of the function f.

Proposition 4.25 ([11]). *The functions defined above have the following properties.*

1. *The function $\hat{f}$ is concave and usc and the function $\check{f}$ is convex and lsc.*
2. *The functions $\hat{f}$ and $\check{f}$ are bounded and $\check{f} \le f \le \hat{f}$.*
3. *$f = \hat{f}$ iff f is concave and usc and $f = \check{f}$ iff f is convex and lsc.*
4. *The mapping $f \mapsto \hat{f}$ is increasing and sublinear, while the mapping $f \mapsto \check{f}$ is increasing and superlinear.*
5. *The mappings $f \mapsto \hat{f}$ and $f \mapsto \check{f}$ are involutions, that is $\hat{\hat{f}} = \hat{f}$ and $\check{\check{f}} = \check{f}$.*
6. *$\widehat{f+g} \le \hat{f} + \hat{g}$, $\quad \widehat{f+g} = \hat{f} + g$ if $g \in \mathrm{Af}(Y)$, $\quad |\hat{f} - \hat{g}| \le \|f-g\|_\infty$, and $\quad \widehat{rf} = r\hat{f} \quad$ for $r > 0$.*

In the case of continuous functions on Y one obtains the following formulae.

Proposition 4.26 ([11]). *Let Y be a compact convex subset of a Hausdorff LCS X and $f \in C(Y)$, a continuous real-valued function. Then*

$$\begin{aligned}\hat{f}(x) &= \sup\{\mu(f) : \mu \in \mathcal{M}^1(Y) \ \text{ and } \ r(\mu) = x\}\\ &= \sup\{\mu(f) : \mu \in \mathcal{M}^1(Y),\ \mu \ \text{ discrete and } \ r(\mu) = x\},\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\check{f}(x) &= \inf\{\mu(f) : \mu \in \mathcal{M}^1(Y) \ \text{ and } \ r(\mu) = x\}\\ &= \inf\{\mu(f) : \mu \in \mathcal{M}^1(Y),\ \mu \ \text{ discrete and } \ r(\mu) = x\}.\end{aligned} \tag{4.12}$$

Let Y be a compact convex subset of a Hausdorff LCS X. Recall that we denote by $\mathcal{M}^+(Y)$ the set of all Borel measures on Y and by $\mathcal{M}^1(Y)$ the set of probability measures. Define an order on $\mathcal{M}^+(Y)$ by

$$\mu \prec \nu \ \overset{\text{def}}{\iff} \ \forall f \in \mathrm{Co}(Y),\ \mu(f) \le \nu(f). \tag{4.13}$$

One says that ν is *more diffuse* than μ and this inequality expresses the fact that the support of ν is more concentrated on the set $\mathrm{ext}(Y)$ of extreme points of the set Y.

Some properties of this order are collected in the following proposition. Recall that one denotes by $r(\mu)$ the barycenter of a probability measure $\mu \in \mathcal{M}^1(Y)$ (see (4.7) and Proposition 4.15).

Proposition 4.27 ([11]). *Let Y be a compact convex subset of a Hausdorff LCS X.*

1. *For every $\mu \in \mathcal{M}^1(Y)$, $\varepsilon_{r(\mu)} \prec \mu$.*
2. *If $\mu, \nu \in \mathcal{M}^+(Y)$ and $\mu \prec \nu$, then $\|\mu\| = \|\nu\|$ and $r(\mu) = r(\nu)$.*
3. *The set $\mathcal{M}^+(Y)$ is inductively ordered, so that every $\mu \in \mathcal{M}^+(Y)$ is majorized by a maximal measure.*
4. *For each $x \in Y$ there exists a maximal measure $\mu \in \mathcal{M}^1(Y)$ with $r(\mu) = x$.*

The set of maximal measures has the following properties.

Proposition 4.28 ([11], P. 27.7). *Let Y be a compact convex subset of a Hausdorff LCS X and $M = \{\mu \in \mathcal{M}^+(Y) : \mu$ is maximal$\}$. Then the following assertions hold true.*

1. *The set M is a convex cone.*
2. *The set M is hereditary on the left, that is*

$$(\nu \in \mathcal{M}^+(Y),\ \mu \in M \text{ and } \nu \le \mu) \implies \nu \in M.$$

3. *The set M is a lattice in its own order $\leq_M$.*

Concerning the supports of maximal measures the following results hold.

Proposition 4.29 ([11], Exercise 26.4 and C. 27.5)**.** *Let Y be a compact convex subset of a Hausdorff LCS X and $\mu \in \mathcal{M}^+(Y)$. Then*

1. *If $S(\mu) \subset \operatorname{ext}(Y)$, then μ is maximal.*
2. *If μ is maximal, then $S(\mu) \subset \operatorname{cl}(\operatorname{ext}(Y))$.*
3. *If, in addition, Y is metrizable, then $S(\mu) \subset \operatorname{ext}(Y)$.*

In general one can prove only the following result.

Theorem 4.30 ([11], Th. 27.11)**.** *Let Y be a compact convex subset of a Hausdorff LCS X and $\mu \in \mathcal{M}^+(Y)$. If μ is maximal, then μ is supported by every $\mathcal{K}$-analytic set containing* $\operatorname{ext}(Y)$.

$\mathcal{K}$-analytic sets are subsets of a topological space obtained from compact sets by taking successive operations of unions and intersections. More exactly, a subset A of a Hausdorff topological space T is $\mathcal{K}$-analytic if there exists a compact Hausdorff space K, a $\mathcal{K}_{\sigma\delta}$-subset B of K and a continuous function $f : B \to T$ s.t. $f(B) = A$. Denote by $\mathcal{K}$ the family of all compact subsets of a topological space T. Then $\mathcal{K}_\sigma$ is the family of all countable unions of sets in $\mathcal{K}$ and $\mathcal{K}_{\sigma\delta}$ is the family of all countable intersections of sets in $\mathcal{K}_\sigma$ (see [10], Section 8, for details).

Let Y be a compact convex subset of a Hausdorff LCS X. For $\mu \in \mathcal{M}^+(Y)$ define $\hat{\mu} : C(Y) \to \mathbb{R}$ by

$$\hat{\mu}(f) = \mu(\hat{f}). \tag{4.14}$$

The definition is correct since $\hat{f}$ is bounded and usc, and so integrable w.r.t. μ. Some properties of this mapping are collected in the following proposition.

Proposition 4.31 ([11])**.** *Let Y be a compact convex subset of a Hausdorff LCS X, $\mu \in \mathcal{M}^+(Y)$ and $\hat{\mu}$ defined by* (4.14)*. Then the following assertions hold true.*

1. *$\mu(f) \leq \hat{\mu}(f)$ for all $f \in C(Y)$ and the mapping $f \mapsto \hat{\mu}(f)$ is sublinear and continuous on $C(Y)$.*
2. *For $\nu \in \mathcal{M}(Y)$,*
$$(\nu \leq \hat{\mu} \text{ on } C(Y)) \iff \big(\nu \in \mathcal{M}^+(Y) \text{ and } \mu \prec \nu\big).$$
3. *$\hat{\mu}$ is linear on $C(Y)$ iff there exists a linear form ν on $C(Y)$ s.t. $\nu \leq \hat{\mu}$.*

Based on these properties one can prove the following result.

Theorem 4.32 ([11], Th. 26.16)**.** *Let Y be a compact convex subset of a Hausdorff LCS X, $\mu \in \mathcal{M}^+(Y)$ and $\hat{\mu}$ defined by* (4.14)*. Then the following assertions are equivalent.*

1. *The measure μ is maximal.*
2. *$\hat{\mu} \in \mathcal{M}^+(Y)$, that is $\hat{\mu}$ is linear on $C(Y)$.*
3. *$\hat{\mu}(f) = \mu(f)$ for every $f \in C(Y)$.*
4. *$\hat{\mu}(f) = \mu(f)$ for every $f \in \operatorname{Co}(Y)$.*

4.7. **Simplexes and uniqueness of representing measures.** The uniqueness of representing measures is closely related to the notion of simplex in infinite dimensional LCS.

A compact convex subset of $\mathbb{R}^n$ is called a *k-simplex* if it is the convex hull of $k+1$ affinely independent points, or equivalently, if it has exactly $k+1$ extreme points. This means that in $\mathbb{R}$ the simplexes are the bounded closed intervals, in $\mathbb{R}^2$ the bounded closed intervals and the triangles, and in $\mathbb{R}^3$ the bounded closed intervals, the triangles and the tetrahedrons.

This definition can be extended to infinite dimensions in the following way.

A nonempty convex subset B of a vector space X is called a *simplex* if

$$\begin{gathered}\forall x, y \in X,\ \forall \alpha, \beta \geq 0, \\ (x+\alpha B) \cap (y+\beta B) = \emptyset \quad \text{or} \quad \exists z \in X,\ \exists \gamma \geq 0,\ (x+\alpha B) \cap (y+\beta B) = z+\gamma B.\end{gathered} \tag{4.15}$$

It is easy to check that for $X = \mathbb{R}^n$ one obtains the usual simplexes, considered at the beginning of this subsection.

A convex subset B of a vector space X is called *linearly compact* if the intersection with each line D in X is empty or a bounded closed interval in D.

Theorem 4.33 (Choquet-Kendall, see [22], T. 3.11)**.** *Let X be a vector space ordered by a generating cone with base B. Then X is a vector lattice iff B is a linearly compact simplex.*

Remark 4.34. In [23] and [11], a simplex is defined as a base for a cone K in a vector space X such that the subspace $Y = K - K$ generated by K is a vector lattice (or equivalently, such that K is a lattice). By Proposition 2.14 this is equivalent to the definition given above.

Simplexes are related to the uniqueness of the representation in the following way.

Theorem 4.35 (Choquet-Meyer, see [11], Th. 28.4)**.** *Let Y be a compact convex subset of a Hausdorff LCS. Then the following assertions are equivalent.*

1. *The set Y is a simplex.*
2. *For every $f \in \mathrm{Co}(Y)$ the function $\hat{f}$ is affine on Y.*
3. *For every maximal measure $\mu \in \mathcal{M}^1(Y)$, $\mu(f) = \hat{f}(r(\mu))$ for all $f \in \mathrm{Co}(Y)$.*
4. *The mapping $f \mapsto \hat{f}$ is linear (meaning additive and positively homogeneous) on* $\mathrm{Co}(Y)$.
5. *For every $x \in Y$ there exists a unique maximal measure $\mu \in \mathcal{M}^1(Y)$ with $r(\mu) = x$.*

References

[1] Ch. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with applications to economics,* 2nd ed., American Mathematical Society (AMS), Providence, RI, 2003.

[2] ______, *Positive operators,* Reprint of the 1985 original, Springer, Berlin, 2006.

[3] Ch. D. Aliprantis and R. Tourky, *Cones and duality*, Graduate Studies in Mathematics, vol. 84, American Mathematical Society, Providence, RI, 2007.

[4] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications,* W. de Gruyter Studies in Mathematics, Vol. 17, W. de Gruyter, Berlin, 1994.

[5] N. Boboc and G. Bucur, *Conuri convexe de functii continue pe spatii compacte* (Romanian), Editura Academiei Republicii Socialiste România, Bucureşti 1976.

[6] N. Boboc and G. Bucur, *Măsură şi capacitate* (Romanian), Editura Ştiinţifică şi Enciclopedică, Bucureşti 1985.

[7] N. Boboc, G. Bucur and A. Cornea, *Order and convexity in potential theory: H-cones*, In collab. with Herbert Höllein, Lecture Notes in Mathematics, Vol. 853, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[8] N. Bourbaki, *Integration* , Springer Verlag, Berlin, 2004
Translated from the French fascicles of Éléments de mathématique, Livre VI, Intégration:
Vol. I. Chapters 1-6 (Fascs. XIII, XXI, XXV);
Vol. II. Chapters 7-9 (Fascs. XXIX, XXXV).
(there are also Russian translations, Moscow 1967 and 1977).[1]

[9] W. W. Breckner, *Rational s-convexity. A generalized Jensen-convexity*, Cluj University Press, Cluj-Napoca, 2011.

[10] G. Choquet, *Lectures on analysis. Vol. I: Integration and topological vector spaces*, Mathematics Lecture Note Series, vol. 24, W.A. Benjamin, Inc., Reading, Mass.,1976, 3rd printing.

[11] ______, *Lectures on analysis, Vol. II: Representation theory.*, Mathematics Lecture Note Series, vol. 25, W.A. Benjamin, Inc., Reading, Mass., 1976, 3rd printing.

[1] Of course, there are the French original versions of these books. Since every time when writing a new chapter the previous ones were revised, I prefered to quote these translations of the actualized versions of the original texts. For instance, Russian translation from 1967 contains Chapters I–V, while the 1977 edition contains the revised chapters III–V and Chapter IX.

[12] ______, *Lectures on analysis, Vol. III: Infinite dimensional measures and problem solutions.*, Mathematics Lecture Note Series, vol. 26, W.A. Benjamin, Inc., Reading, Mass., 1976, 3rd printing.
[13] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.
[14] K. Donner, *Extension of positive operators and Korovkin theorems,* Lecture Notes in Mathematics, Vol. 904, Springer-Verlag, Berlin-Heidelberg-New York, 1982.
[15] J. Elstrodt, *Mass- und Integrationstheorie*, Springer-Lehrbuch, Springer-Verlag, Berlin 2011 (7 Auflage).
[16] D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge University Press, Cambridge 1974,
[17] D. Guo, Y. J. Cho, and J. Zhu, *Partial ordering methods in nonlinear problems*, Nova Science Publishers Inc., Hauppauge, NY, 2004.
[18] D. H. Hyers, G. Isac, and Th. M. Rassias, *Topics in nonlinear analysis & applications*, World Scientific Publishing Co. Inc., River Edge, NJ, 1997.
[19] G. Jameson, *Ordered linear spaces*, Lecture Notes in Mathematics, Vol. 141, Springer-Verlag, Berlin, 1970.
[20] G. Köthe, *Topological vector spaces I*, Springer Verlag, Berlin-Heidelberg-New York, 1969.
[21] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.
[22] A. L. Peressini, *Ordered topological vector spaces*, Harper & Row Publishers, New York, 1967.
[23] R. R. Phelps, *Lectures on Choquet's theorem*, Lecture Notes in Mathematics, vol. 1757, Springer, Berlin, 2001.
[24] M. D. Rus, *The method of monotone iterations for mixed monotone operators*, Babes-Bolyai University, Ph.D. Thesis, Cluj-Napoca, 2010.
[25] H. H. Schaefer, *Topological vector spaces*, Third printing corrected, Graduate Texts in Mathematics, Vol. 3, Springer-Verlag, New York, 1971.
[26] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.