Characterizations of $\varepsilon$-duality gap statements for composed optimization problems

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Consider two separated locally convex vector spaces $X$ and $Y$ and their continuous dual spaces $X^*$ and $Y^*$, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$ respectively. Let the nonempty closed convex cone $C \subseteq Y$ and its dual cone $C^* = \{ y^* \in Y^* : \langle y^*, y \rangle \geq 0 \ \forall y \in Y \}$.

Let $f : X \to \overline{\mathbb{R}}$ be a proper function, $g : Y \to \overline{\mathbb{R}}$ be a proper function, which is also $C$-increasing and $h : X \to Y^*$ be a proper vector function fulfilling $\text{dom} g \cap (h(\text{dom} f) + C) \neq \emptyset$. Unless otherwise stated, these hypotheses remain valid through the entire chapter. Consider the optimization problem

$$\inf_{x \in X} [f(x) + (g \circ h)(x)]. \quad (P^C)$$

For $x^* \in X^*$ we also consider the linearly perturbed optimization problem

$$\inf_{x \in X} [f(x) + (g \circ h)(x) - \langle x^*, x \rangle]. \quad (P^C_{x^*})$$
To this problem we can attach different dual Fenchel-Lagrange-type problems. If $f$ and $(\lambda h)$ are taken together one gets the following dual to $(P^C_{x^*})$

$$\sup_{\lambda \in C^*} \{-g^*(\lambda) - (f + (\lambda h))^*(x^*)\}.$$  \hspace{1cm} (D^C_{x^*})$$

When $f$ and $(\lambda h)$ are separated, one gets the following dual problem

$$\sup_{\lambda \in C^*, \beta \in X^*} \{-g^*(\lambda) - f^*(\beta) - (\lambda h)^*(x^* - \beta)\}.$$  \hspace{1cm} (\overline{D^C_{x^*}})$$
**ε-duality gap statements using epigraphs**

Let $\varepsilon \geq 0$. Consider the regularity conditions

\[
\{(x^*, 0, r) : (x^*, r) \in \text{epi}(f + g \circ h)^*\} \subseteq \left[\{0\} \times \text{epi}(g^*) + \bigcup_{\lambda \in C^*} \{(a, -\lambda, r) : (a, r) \in \text{epi}((f + (\lambda h))^*)\} \right] \cap (X^* \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon)
\]

and

\[
\{(x^*, 0, r) : (x^*, r) \in \text{epi}(f + g \circ h)^*\} \subseteq \left[\{0\} \times \text{epi}(g^*) + \left\{(x^*, 0, r) : (x^*, r) \in \text{epi}(f^*)\right\} \bigcup_{\lambda \in C^*} \{(a, -\lambda, r) : (a, r) \in \text{epi}((\lambda h)^*)\}\right] \cap (X^* \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon)
\]
(H.-V. Boncea, S.-M. Grad, [1]) The condition (RC) is fulfilled if and only if for any \( x^* \in X^* \) there exists a \( \lambda \in C^* \) such that

\[
(f + g \circ h)^* (x^*) \geq g^*(\lambda) + (f + (\lambda h))^* (x^*) - \varepsilon. \tag{1}
\]

Remark

In the left-hand side of (1) one can easily recognize \(-\nu(P_{x^*}^C)\). The quantity in the right-hand side of (1) is not necessarily \(-\nu(D_{x^*}^C) - \varepsilon\), as the supremum in \((D_{x^*}^C)\) is not shown to be attained at \(\lambda\). Though, (1) implies \(\nu(P_{x^*}^C) \leq \nu(D_{x^*}^C) + \varepsilon\), which actually means that for \((P_{x^*}^C)\) and \((D_{x^*}^C)\) there is \(\varepsilon\)-duality gap. Thus, (RC) yields that there is stable \(\varepsilon\)-duality gap for \((P_{x^*}^C)\) and \((D_{x^*}^C)\). Note also that \(\lambda \in C^*\) obtained in the above theorem is an \(\varepsilon\)-optimal solution of \((D_{x^*}^C)\).
Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition \((RC)\) is fulfilled if and only if for any \(x^* \in X^*\) there exist some \(\lambda \in C^*\) and \(\beta \in X^*\) such that

\[
(f + g \circ h)^*(x^*) \geq g^*(\lambda) + f^*(\beta) + (\lambda h)^*(x^* - \beta) - \varepsilon. \tag{2}
\]

Remark

In the left-hand side of (2) one can easily recognize \(-v(P^C_{x^*})\). The quantity in the right-hand side of (2) is not necessarily \(-v(D^C_{x^*}) - \varepsilon\), as the supremum in \((D^C_{x^*})\) is not shown to be attained at \(\lambda\) and \(\beta\). Though, (2) implies \(v(P^C_{x^*}) \leq v(D^C_{x^*}) + \varepsilon\), which actually means that for \((P^C_{x^*})\) and \((D^C_{x^*})\) there is \(\varepsilon\)-duality gap. Thus \((RC)\) guarantees stable \(\varepsilon\)-duality gap for \((P^C)\) and \((D^C)\) and, moreover, also for \((P^C)\) and \((D^C)\). Note also that the pair \((\lambda, \beta) \in C^* \times X^*\) obtained in the above theorem is an \(\varepsilon\)-optimal solution of \((D^C_{x^*})\).
In order to characterize formulae similar to (1) and (2), where appear actually the optimal values of \((D^C)\) and \((\overline{D^C})\), let us consider the following regularity conditions

\[
epi(f + g \circ h)^* \subseteq \epi \inf_{\lambda \in C^*} [g^*(\lambda) + (f + (\lambda h))^*(\cdot)] - (0, \varepsilon) \quad (RCl)
\]

and

\[
epi(f + g \circ h)^* \subseteq \epi \inf_{\lambda \in C^*} [g^*(\lambda) + f^*(\beta) + (\lambda h)^*(\cdot - \beta)] - (0, \varepsilon). \quad (\overline{RCl})
\]
Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition (RCI) is fulfilled if and only if for any \( x^* \in X^* \) we have

\[
(f + g \circ h)^*(x^*) \geq \inf_{\lambda \in C^*} \left[ g^*(\lambda) + (f + (\lambda h))^*(x^*) \right] - \varepsilon. \tag{3}
\]

Remark

Relation (3) means actually \( v(P^C_{x^*}) \leq v(D^C_{x^*}) + \varepsilon \), i.e. we have stable \( \varepsilon \)-duality gap for \( (P^C) \) and \( (D^C) \).

Theorem

(H.-V. Boncea, S.-M. Grad, [1]) The condition (RCI) is fulfilled if and only if for any \( x^* \in X^* \) we have

\[
(f + g \circ h)^*(x^*) \geq \inf_{\lambda \in C^*} \left[ g^*(\lambda) + f^*(\beta) + (\lambda h)^*(x^* - \beta) \right] - \varepsilon. \tag{4}
\]
\(\varepsilon\)-duality gap statements using subdifferentials

**Theorem**

(H.-V. Boncea, S.-M. Grad, [1]) One has

\[
\partial(f + g \circ h)(x) \subseteq \bigcap_{\eta > 0} \bigcup_{\varepsilon_1, \varepsilon_2 \geq 0} \partial_{\varepsilon_1}(f + (\lambda h))(x) \quad (RCSC)
\]

for all \( x \in X \) if and only if (3) holds for all \( x^* \in R(\partial(f + g \circ h)) \).

**Theorem**

(H.-V. Boncea, S.-M. Grad, [1]) One has

\[
\partial(f + g \circ h)(x) \subseteq \bigcup_{\varepsilon_1, \varepsilon_2 \geq 0} \partial_{\varepsilon_1}(f + (\lambda h))(x) \quad (RCLC)
\]

for all \( x \in X \) if and only if for all \( x^* \in R(\partial(f + g \circ h)) \), there exists \( \lambda \in C^* \) such that (1) holds.
(H.-V. Boncea, S.-M. Grad, [1]) One has

$$\partial (f + g \circ h)(x) \subseteq \bigcap_{\eta > 0} \bigcup_{\varepsilon_1, \varepsilon_2 \geq 0} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (\lambda h)(x) \quad (\text{RCSC})$$

for all $x \in X$ if and only if for all $x^* \in \mathcal{R}(\partial (f + g \circ h))$, (4) holds.

(H.-V. Boncea, S.-M. Grad, [1]) One has

$$\partial (f + g \circ h)(x) \subseteq \bigcup_{\varepsilon_1, \varepsilon_2 \geq 0} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (\lambda h)(x) \quad (\text{RCLC})$$

for all $x \in X$ if and only if for all $x^* \in \mathcal{R}(\partial (f + g \circ h))$, there exist

$$\overline{\lambda} \in C^* \text{ and } \overline{\beta} \in X^*$$

such that (2) holds.
Results concerning $\varepsilon$-optimality conditions, $\varepsilon$-Farkas statements and $(\varepsilon, \eta)$-saddle points

From the results presented in the previous sections one can derive other useful statements concerning $\varepsilon$-optimality conditions, $\varepsilon$-Farkas assertions and characterizations for $(\varepsilon, \eta)$-saddle points. Let us consider the following regularity conditions:

\[
\left( \text{epi}(f + g \circ h)^* \right) \cap \left( \{0\} \times \mathbb{R} \right) \subseteq \left( \text{epi} \inf_{\lambda \in C^*} \left[ g^*(\lambda) + (f + (\lambda h))^*(\cdot) \right] \right) \cap \left( \{0\} \times \mathbb{R} \right) - (0, \varepsilon)
\]

\((RCI^0)\)

and

\[
\left( \text{epi}(f + g \circ h)^* \right) \cap \left( \{0\} \times \mathbb{R} \right) \subseteq \left( \text{epi} \inf_{\lambda \in C^*} \left[ g^*(\lambda) + f^*(\beta) + (\lambda h)^*(\cdot - \beta) \right] \right) \cap \left( \{0\} \times \mathbb{R} \right) - (0, \varepsilon).
\]

\((\overline{RCI}^0)\)
(H.-V. Boncea, S.-M. Grad, [1])

(a) Let $\varepsilon, \eta \geq 0$. Suppose that the condition $(RCI^0)$ is fulfilled. If $\overline{x}$ is an $\varepsilon$-optimal solution of the problem $(P^C)$, then there exist $\varepsilon_1, \varepsilon_2 \geq 0$, and $\overline{\lambda} \in C^*$ such that

(i) $g^*(\overline{\lambda}) + g(h(\overline{x})) \leq (\overline{\lambda} \cdot h)(\overline{x}) + \varepsilon_2$,

(ii) $f^*(\overline{\lambda} \cdot h)(0) + (f + (\overline{\lambda} \cdot h))(\overline{x}) \leq \varepsilon_1$,

(iii) $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta$.

Moreover, $\overline{\lambda}$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(D^C)$.

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$ and $\overline{\lambda} \in C^*$ such that the relations (i)-(iii) hold for $\overline{x} \in X$ and $\overline{\lambda} \in C^*$, then $\overline{x}$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(P^C)$. Moreover, $\overline{\lambda}$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(D^C)$. 
The similar statement for \( (\overline{D}^C) \) can be proven analogously.

**Theorem**

(H.-V. Boncea, S.-M. Grad, [1]) (a) Let \( \varepsilon, \eta \geq 0 \). Suppose that the condition \( (\overline{RCI^0}) \) is fulfilled. If \( \overline{x} \) is an \( \varepsilon \)-optimal solution of the problem \( (P^C) \), then there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \overline{\lambda} \in C^* \) and \( \overline{\beta} \in X^* \) such that

(i) \( g^*(\overline{\lambda}) + g(h(\overline{x})) \leq (\overline{\lambda}h)(\overline{x}) + \varepsilon_3 \),
(ii) \( f^*(\overline{\beta}) + f(\overline{x}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1 \),
(iii) \( (\overline{\lambda}h)^*(-\overline{\beta}) + (\overline{\lambda}h)(\overline{x}) \leq \langle -\overline{\beta}, \overline{x} \rangle + \varepsilon_2 \),
(iv) \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \eta \).

Moreover, \( (\overline{\lambda}, \overline{\beta}) \) is an \( (\varepsilon + \eta) \)-optimal solution of the problem \( (\overline{D}^C) \).

(b) If there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \overline{\lambda} \in C^* \) and \( \overline{\beta} \in X^* \) such that the relations (i)-(iv) hold for \( \overline{x} \in X, \overline{\lambda} \in C^* \) and \( \overline{\beta} \in X^* \) then \( \overline{x} \) is an \( (\varepsilon + \eta) \)-optimal solution of the problem \( (P^C) \). Moreover, \( (\overline{\lambda}, \overline{\beta}) \) is an \( (\varepsilon + \eta) \)-optimal solution of the problem \( (\overline{D}^C) \).
In the following we give $\varepsilon$-Farkas-type results for $(P^C)$ and its duals, too. Consider the following conditions:

$$\{(0, 0, r) : (0, r) \in epi(f + g \circ h)^*\} \subseteq \{0\} \times epi(g^*) + \bigcup_{\lambda \in C^*} \{(a, -\lambda, r) : (a, r) \in epi((f + (\lambda h))^*)\} \cap (\{0\} \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon)$$

$(RC^0)$

and

$$\{(0, 0, r) : (0, r) \in epi(f + g \circ h)^*\} \subseteq \{0\} \times epi(g^*) + \{(0, 0, r) : (0, r) \in epi(f^*)\} + \bigcup_{\lambda \in C^*} \{(a, -\lambda, r) : (a, r) \in epi((\lambda h)^*)\} \cap (\{0\} \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon)$$

$(\overline{RC^0})$

**Theorem**

(i) Suppose that $(RC^0)$ holds. If $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$ then there exists $\overline{\lambda} \in C^*$ such that $g^*(\overline{\lambda}) + (f + \overline{\lambda} h)^*(0) \leq \varepsilon/2$.

(ii) If there exists $\overline{\lambda} \in C^*$ such that $g^*(\overline{\lambda}) + (f + \overline{\lambda} h)^*(0) \leq -\varepsilon/2$, then $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$. 
Analogously, one can prove the following statements for \((P^C)\) and \((D^C)\), too.

**Theorem**

(i) Suppose that \((RC^0)\) holds. If \(f(x) + (g \circ h)(x) \geq \varepsilon/2\) for all \(x \in X\) then there exist \(\lambda \in C^*\) and \(\beta \in X^*\) such that

\[
f^*(\beta) + g^*(\lambda) + (\lambda h)^*(-\beta) \leq \varepsilon/2.
\]

(ii) If there exist \(\lambda \in C^*\) and \(\beta \in X^*\) such that

\[
f^*(\beta) + g^*(\lambda) + (\lambda h)^*(-\beta) \leq -\varepsilon/2,
\]

then \(f(x) + (g \circ h)(x) \geq \varepsilon/2\) for all \(x \in X\).
Nevertheless, one can extend the investigations from this section also towards generalized saddle points.

The Lagrangian function assigned to \((P^C) - (D^C)\) is \(L^C : X \times Y^* \rightarrow \overline{\mathbb{R}}\), defined by (cf. [5])

\[
L^C(x, \lambda) = \begin{cases} 
    f(x) + (\lambda h)(x) - g^*(\lambda), & \text{if } \lambda \in C^* \\
    -\infty, & \text{otherwise.}
\end{cases}
\]

Let \(\eta \geq 0\). We say that \((\overline{x}, \overline{\lambda}) \in X \times Y^*\) is \((\eta, \varepsilon)\)-saddle point of the Lagrangian \(L^C\) if

\[
L^C(\overline{x}, \lambda) - \eta \leq L^C(\overline{x}, \overline{\lambda}) \leq L^C(x, \overline{\lambda}) + \varepsilon, \text{ for all } (x, \lambda) \in X \times Y^*.
\]

**Theorem**

(H.-V. Boncea, S.-M. Grad, [1]) Assume that \(g\) is a convex and lower semicontinuous function fulfilling \(g(y) > -\infty\) for all \(y \in Y\). If \((\overline{x}, \overline{\lambda})\) is an \((\eta, \varepsilon)\)-saddle point of \(L^C\) then \(\overline{x} \in X\) is an \((\varepsilon + \eta)\)-optimal solution to \((P^C)\), \(\overline{\lambda} \in C^*\) is an \((\varepsilon + \eta)\)-optimal solution to \((D^C)\) and there is \((\varepsilon + \eta)\)-duality gap for the pair of problems \((P^C)\) and \((D^C)\), i.e.

\[\nu(P^C) \leq (D^C) + \varepsilon + \eta.\]
An analogous result with the anterior theorem can be formulated for the pair of problems \((P^C)\) and \((D^C)\) with the corresponding Lagrangian function given by (cf. [5])

\[
\overline{L}^C : X \times X^* \times Y^* \to \mathbb{R}
\]

\[
\overline{L}^C(x, \beta, \lambda) = \begin{cases} 
\langle \beta, x \rangle + (\lambda h)(x) - f^*(\beta) - g^*(\lambda), & \text{if } \lambda \in C^* \\
-\infty, & \text{otherwise}.
\end{cases}
\]

**Theorem**

Assume that \(g\) is a convex and lower semicontinuous function fulfilling \(g(y) > -\infty\) for all \(y \in Y\). If \((\overline{x}, \overline{\lambda})\) is an \((\eta, \varepsilon)\)-saddle point of \(\overline{L}^C\) then \(\overline{x} \in X\) is an \((\varepsilon + \eta)\)-optimal solution to \((P^C)\), \(\overline{\lambda} \in C^*\) is an \((\varepsilon + \eta)\)-optimal solution to \((D^C)\) and there is \((\varepsilon + \eta)\)-duality gap for the pair of problems \((P^C)\) and \((D^C)\), i.e. \(\nu(P^C) \leq (D^C) + \varepsilon + \eta\).
Bibliography


Vă mulțumesc pentru atenție!