Closed convex sets with an open or closed Gauss range

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29.10.2020

Abstract

The Gauss map of a closed convex set C, as defined by Laetsch, generalizes the Gauss map of an orientable regular hypersurface of the n-dimensional Euclidean space. While the shape of such a regular hypersurface is well encoded by the Gauss map, the range of this map, equally called the spherical image of the hypersurface, is used to study various aspects of the smooth convex hypersurfaces both in the finite dimensional setting (doCarmo-Lawson, Wu) and in the infinite dimensional setting (deAndrade, Laetsch) as well. The study of closed convex sets through their Gauss ranges encounters an interior versus boundary dichotomy for the points of the Gauss range, which is reflected in a bounded versus unbounded dichotomy at the facial structure level. Indeed, every interior point of the Gauss range produces an exposed bounded proper face and every intersection point of the Gauss range with its boundary produces an unbounded proper exposed face of the involved closed convex set, whenever this closed convex set has nonempty interior. On the other hand it is self-evident that the lack of border points in the Gauss range, which happens when the Gauss range is open, entails the lack of exposed unbounded proper faces. These two situations, i.e. open versus closed for the Gauss ranges, are extreme for every size evaluation tool of the intersection between the Gauss range and its boundary. This intersection set is expected to encode the amount of unbounded exposed proper faces. At one extreme we have those unbounded closed convex sets with several unbounded proper faces. An important class of such unbounded closed convex sets are the two or higher dimensional Motzkin decomposable ones, as their Gauss ranges happen to be closed. The other extreme is represented by the unbounded closed convex sets with nonempty interior and compact proper faces, as their Gauss ranges are open. Such unbounded closed convex sets are Minkowski sets. The main purpose of this paper is to characterize the closed convex subsets of the n-dimensional Euclidean space that have an open or a closed Gauss range. We will emphasize the case of epigraphs of lower semicontinuous proper convex functions. While the characterizations of general closed convex subsets of the n-dimensional Euclidean space with open or closed Gauss ranges have some geometric flavor, in the particular case of epigraphs they become more abstract. All these characterizations employ fundamental concepts and results of Convex Analysis, such as the normal, the barrier and the recession cones of a convex subset C of the n-dimensional Euclidean space as well as its supporting hyperplanes. In the case of epigraphs of lower semicontinuous convex functions, our characterizations involve conjugates and subdifferential operators.