Finite Functions in $C(Q)$ and Finite Elements in Vector Lattices

Martin R. Weber
Department of Mathematics, Technical University Dresden
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Abstract

Some detailed analysis of continuous functions with compact support (i.e. finite functions) leads to the notion of finite, totally finite and self-majorizing elements in a vector lattice $E$, e.g. an element $\varphi \in E$ is called finite if there exists an element $z \in E$ (a majorant of $\varphi$) satisfying the property: for each $x \in E$ there is a number $c_x > 0$ such that the inequality

$$|x| \wedge n|\varphi| \leq c_x z$$

holds for any $n \in \mathbb{N}$.

For a vector sublattice $H \subset E$ the relations between finite elements in $H$ and in $E$ are studied. In general, even a finite element in $H$ may not be finite in $E$.

If the set $\mathcal{M}(E)$ of all maximal ideals of an Archimedean vector lattice $E$ is equipped with the hull-kernel topology $\tau_{hk}$ then the topological space $(\mathcal{M}(E), \tau_{hk}) =: \mathcal{M}(E)$ turns out to be a Hausdorff space but, in general, does not satisfy any stronger separation axiom. The space $\mathcal{M}(E)$ carries many information on the vector lattice $E$.

The element $\varphi$ is called totally finite, if it possesses a majorant which itself is a finite element. A finite element $\varphi$ is called selfmajorizing, if $|\varphi|$ is a majorant of $\varphi$.

Finite, totally finite and selfmajorizing elements of a vector lattice $E$ can be characterized by means of their abstract supports, i.e. the $\tau_{hk}$-closure of subsets $G_x := \{ M \in \mathcal{M}(E) : x \notin M \}$ which are defined in $\mathcal{M}(E)$ for any $x \in E$.

The $\tau_{hk}$-closure supp$_{\mathcal{M}}(x)$ of $G_x$ is the so-called abstract support of the element $x$. In radical-free vector lattices a finite element $\varphi$ is characterized by the compactness of its abstract support and the majorants $z$ of $\varphi$ by the inclusion supp$_{\mathcal{M}}(\varphi) \subset G_z$. Totally finite and selfmajorizing elements can be similarly characterized.

The subspace $\mathcal{M}_{\Phi}(E) = \bigcup \{ G_\varphi : \varphi \text{ is a finite element in } E \}$ plays an important role for the representation of vector lattices by means of continuous functions, namely in the situation if one asks for representations, where the isomorphic image of each finite element is a functions with compact support.

Literature: