Parametric vector optimization

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Content

Results:

- $(EP)_n$, (EP);
- *Parametric vector optimization*.

Definitions and condition:

- Brézis topological pseudomonotonicity (for bifunction);
- Mosco convergence;
- Condition (C).

Applications:

- abstract equilibrium problems; variational inequalities;
- parametric optimization; variational calculus;
- Walras equilibrium.

  Bibliography

Basics

(EP) find $a \in X$ such that

$$f(a, b) \geq 0, \text{ for all } b \in X,$$

where $f : X \times X \to R$ is a given function.
Particular cases:

- $f = -|g|$ neutral element;
- $f(a, b) = g(b) - g(a)$ optimization;
- $f(a, b) = \langle g(a), b - a \rangle$ variational inequality; Fermat's theorem $g(a) = \nabla l(a)$ on $[a, b] \subset R^m$.

(EP), $(EP)_n$

Let (X, σ) be a Hausdorff topological space. Let D be a nonempty subset of X.
The scalar equilibrium problem supposes to find an element $a \in D$ such that $f(a, b) \geq 0$, for all $b \in D$, where $f : X \times X \to R$ is a given function.
For each $n \in N$, the (parametric) equilibrium problem is the following:
$(EP)_n$ find $a_n \in D_n$ such that

$$f_n(a_n, b) \geq 0, \text{ for all } b \in D_n,$$

where D_n is a nonempty subset of X and $f_n : X \times X \to R$ is a given function.

(P), $(P)_n$

(Q) If $(a_n)_n$ is a sequence of solutions for a sequence of problems $(P)_n$ and $a_n \to x$ in a topological space X is it true that x is a solution for a problem P ?

$(EP)_n$, (EP); $(VEP)_n$, (VEP). Same recipe for $(PO)_n$, (PO).

Motivation

As a result of changes in the problem data, the solutions behavior is always of concern. For instance, a sequence of functions may provide a sequence of solutions, therefore we are interested to study a certain **stability of this sequence**.

Formalism

Let X be a topological space.

$(P)_n$ $\qquad$ (P)

$a_n \in S(n), a_n \to a$ $\qquad$ $a \in S(\infty)$?

Denote by $S(n)$ the set of the solutions for $(P)_n$ (n fixed) and by $S(\infty)$ the set of the solutions for (P).

Denote by $S(n)$ the set of the solutions for $(EP)_n$ (n fixed) and by $S(\infty)$ the set of the solutions for (EP).

Theorem

[Bogdan-Kolumbán, TMNA 2012]

Let X be a Hausdorff topological space with σ and τ topologies on X such that $\sigma \subseteq \tau$, i.e. σ is weaker than τ. Suppose that $S(n) \neq \emptyset$, for each $n \in N$, and the following conditions hold:

- *conditions on Φ_n, Φ; (in particular parametric domains, Mosco)*
- *condition that relates f_n and the limit function f;*
- *the property on the limit function f.*

Then, for each sequence $(a_n)_{n \in N}$ with $a_n \in S(n)$, $a_n \stackrel{\sigma}{\to} a$ implies $a \in S(\infty)$.

Parametric domains. Mosco convergence

Definition

([Mosco]) Let X be a Banach space and $U \subseteq X$. A sequence of sets (U_n) in X is Mosco convergent to U ($U_n \xrightarrow{M} U$) if

$$w - \mathrm{Limsup}\, U_n \subseteq U \subseteq s - \mathrm{Liminf}\, U_n.$$

In the definition above, $w - \mathrm{Limsup}\, U_n$ denotes the set of all the points v such that $v_k \rightharpoonup v$, with $v_k \in U_{n_k}$, for all k and (U_{n_k}), n_k subsequence and $s - \mathrm{Liminf}\, U_n$ denotes the set of all the points v such that $v_n \to v$, with $v_n \in U_n$, for n sufficiently large.

Condition (C)

The functions $f_n, f : X \times X \to R$ $(n \in N)$ verify condition:
(**C**) For each sequences $(a_n)_{n \in N}$ and $(b_n)_{n \in N}$ with $a_n \in S(n)$, $a_n \overset{\sigma}{\to} a$, and $b_n \overset{\tau}{\to} b$, one has

$$\liminf_n \big(f(a_n, b) - f_n(a_n, b_n) \big) \geq 0.$$

Pseudomonotonicity in the sense of Brézis

Definition

([AUBIN], pg. 410) A function $f : X \times X \to R$ is said to be topologically pseudomonotone w.r.t. the first variable if, for each sequence $(a_n)_{n\in N} \subset X$ with $a_n \stackrel{\sigma}{\to} a$ in X, $\liminf_n f(a_n, a) \geq 0$ implies

$$\limsup_n f(a_n, b) \leq f(a, b), \text{ for all } b \in X.$$

The case $f(a, b) = g(b) - g(a)$.

We take $f(a,b) = g(b) - g(a)$ and $f_n(a,b) = g_n(b) - g_n(a)$.
If g is lower semi-continuous, then f is, obviously, topologically pseudomonotone w.r.t. the first variable. In this case condition $(\mathbf{C}')$ becomes:
$(\mathbf{C}'')$ For each sequence $(a_n)_{n\in\mathbf{N}}$ of solutions for $(M)_n$, $a_n \to a$ and $b \in X$, there exists a sequence $(b_n)_{n\in\mathbf{N}}$ such that $b_n \to b$ and

$$\liminf_n [g(b) - g_n(b_n) - g(a_n) + g_n(a_n)] \geq 0.$$

Corollary

Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of solutions for $(M)_n$ and let $a_n \to a$. Suppose that g is lower semi-continuous at a and the functions g_n, g, $n \in \mathbf{N}$, verify condition $(\mathbf{C}'')$.
Then, limit a is solution for (M).

How this result can be formulated for vector functions ?

(VEP), $(VEP)_n$, Pareto optimization

For vector problem there exists the following model. Let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, nonempty convex closed in $\mathcal{Z}$, different from $\mathcal{Z}$. For $n \in N$ consider the following vector equilibrium problem:
$(VEP)_n$ find $a_n \in D_n$ such that

$$h_n(a_n, b) \in (-\mathrm{Int}\ C)^c, \text{ for all } b \in D_n,$$

where D_n is a nonempty subset of X and $h_n : X \times X \to \mathcal{Z}$ is given. By $(-\mathrm{Int}\ C)^c$ the complementary of $(-\mathrm{Int}\ C)$ in $\mathcal{Z}$ is denoted. Note that if $\mathcal{Z} = R$ and $C = [0, +\infty)$, then the vector equilibrium problem reduces to scalar equilibrium problem.

Pareto optimization

The parametric generalized optimization (weak) Pareto problem is considered the following:
$(PO)_n$ find $a_n \in D_n$ such that

$$\varphi_n(b) - \varphi_n(a_n) \in (-\mathrm{Int}\, C)^c, \text{ for all } b \in D_n,$$

where $\varphi_n : X \to \mathcal{Z}$ is a given function.
Let $S(n)$ be the set of solutions for $(PO)_n$ and let $S(\infty)$ be the solutions set of (PO).
(Q) If $a_n \in S(n)$ and $a_n \to x$ in X when $n \to \infty$, is it true that $x \in S(\infty)$?

Vector topological pseudomonotonicity

Definition

([Salamon-Bogdan], 2010) Let (X, σ) be a Hausdorff topological vector space and let $D \subseteq X$ be nonempty. We say that a function $h : D \times D \to \mathcal{Z}$ is vector top. pseudomonotone if for all $b \in D$, $v \in \mathrm{Int}\, C$ and for each sequence $(a_n)_{n \in N}$ in D with $a_n \xrightarrow{\sigma} a$ and

$$\mathrm{Liminf}\ h(a_n, a) = \emptyset \quad \text{or} \quad \mathrm{Liminf}\ h(a_n, a) \cap (-\mathrm{Int}\, C)^c \neq \emptyset$$

there exists an index n_0 such that

$$\overline{\{h(a_m, b) : m \geq n\}} \subset h(a, b) + v - \mathrm{Int}\, C, \ \text{for all } n \geq n_0.$$

Condition (C) vector case

The functions $f_n, f : X \times X \to \mathcal{Z}$ $(n \in N)$ verify the following vector condition:
(**VC**) For each sequence $(a_n)_{n \in N}$ with $a_n \in S(n)$, $a_n \to a$, there exists $(b_n)_{n \in N}$ with $b_n \to b$ such that

$$\mathrm{Liminf}\Big(f(b) - f(a_n) - f_n(b_n) + f_n(a_n) \Big) \cap C \neq \emptyset.$$

Result for (PO)

Theorem

[Salamon-Bogdan, JMAA 2010] *Let X be a Hausdorff topological space. Let $(a_n)_{n\in N}$ be such that a_n is a Pareto optima for $(PO)_n$ and let $a_n \to \bar{a}$ in X.*
Suppose that vector condition (**VC**) *applies. If φ is $C-$lower semi-continuous at $\bar{a}$, then $\bar{a}$ is a solution for (PO).*

A function $\varphi : X \to \mathcal{Z}$ is said to be $C-$lower semicontinuous at a if for all $v \in \operatorname{Int} C$ and for each sequence $(a_n)_{n\in N}$ with $a_n \to a$ there exists an index n_0 such that

$$\overline{\{\varphi(a_m) : m \geq n\}} \subset \varphi(a) - v + \operatorname{Int} C, \text{ for all } n \geq n_0.$$

$C-$semi-continuity

Definition

(Tanaka, 1997) A function $\varphi : X \to \mathcal{Z}$ is said to be $C-$lower semi-continuous on X if for every $z \in \mathcal{Z}$ the set $f^{-1}(z + \mathrm{Int}\, C)$ is open in X.

Definition

(Corley, 1980) Let C be a cone in $\mathcal{Z}$. A function $\varphi : X \to \mathcal{Z}$ is said to be $C-$semi-continuous on X if for every $y \in \mathcal{Z}$ the set $f^{-1}(y + \mathrm{cl}\, C)$ is closed in X.

We are able to find the stability of solutions in the presence of vector condition (**VC**) and $C-$lower semi-continuity (in the Tanaka's sense) but we have no result so far on this issue with

Strong $C-$semi-continuity

Definition

(Oppezzi-Rossi, 2006) A function $f : X \to \mathcal{Z}$ is said to be strongly lower $C-$semi-continuous at the point $a \in X$ iff, for any $\varepsilon \in \mathrm{Int}\, C$, there exists $U_{a\varepsilon}$, a neighborhood of a such that

$$f(x) \in f(a) - \varepsilon + C_0, \text{ for all } x \in U_{a\varepsilon}.$$

Example

Example

(Oppezzi-Rossi; Jota, 2006) Let $f : R \to R^2$ be given by

$$f(a) = \begin{cases} (a, 1/a), & \text{if } a > 0, \\ (a, -a^2), & \text{if } a \leq 0 \end{cases}$$

and $f_n = f$, $n \in N$. Let

$C = \{(x, y) \in R^2 : 0 \leq y \leq x\}$.

The function f is not strongly $C-$lower semi-continuous at $a = 0$. Although it is easy, let us proceed. There exists $\varepsilon = (1/2, 1/4) \in \mathrm{Int}\, C$ such that for every U neighborhood of 0, we can find $x_U \in U$ such that

$$f(x_U) + \varepsilon \notin C_0.$$

Example

Let U be a neighborhood of 0. There exists $0 < r < 1$ such that $(-r, r) \subseteq U$. Let us take $x_U = r/2$. We obtain $(r/2, 2/r) + (1/2, 1/4) \notin C_0$. Now, let us consider the sequence $(a_n)_{n\in N}$, $a_n = 1/n$ that are global minimum points so they are also weak minimums for f. Condition (**VC**) applies since, for each $b \in X$ there exists a sequence $(b_n)_{n\in N}$, $b_n \to b$ such that

$$\mathrm{Liminf}\left[f(b) - f(a_n) - f(b_n) + f(a_n)\right] \cap C \neq \emptyset.$$

Indeed, take $b_n = b$, $n \in N$, so $(0, 0)$ is the common element. Observe that 0 is not a weak minimum for f. Straight from the definition one has $(1/2, 1/4) \in [(0, 0) - f(R)] \cap \mathrm{Int}\, C \neq \emptyset$, i.e. there exists $b = -1/2 \in R$ such that

$$f(b) - f(0) \notin (-\mathrm{Int}\, C)^c.$$

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Thank you !

Vă mulțumesc !

Întrebare din anul 2008

Care să fie condițiile impuse asupra unei funcții $f : X \times X \to R$ încât subdiferențiala sa $\partial f : X \times X \to 2^R$ să fie topologic pseudomonotonă ?

Set-valued pseudomonotone operator

Definition

An operator $\mathcal{A} : X \to 2^{X^*}$ is said to be topologically pseudomonotone if the following three conditions hold:

(i) the set $\mathcal{A}u$ is nonempty, bounded, closed and convex for all $u \in X$;

(ii) $\mathcal{A}$ is upper semicontinuous from the segments of X to the weak topology on X^*;

(iii) if $(u_i) \subset X$ with $u_i \rightharpoonup u$ in X and $u_i^* \in \mathcal{A}u_i$ is such that $\liminf_i \langle u_i^*, u - u_i \rangle_X \geq 0$, then to each element $v \in X$ there exists $u^* \in \mathcal{A}u$ to satisfy $\limsup_i \langle u_i^*, v - u_i \rangle_X \leq \langle u^*, v - u_i \rangle_X, \forall v \in X$.