FIXED POINT STRUCTURE THEORY IN METRIC SPACES

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Introduction and preliminaries

The aim of this paper is to present some open problems in the fixed point theory in terms of fixed point structures. To achieve our aim we recall a few basic notions and examples of the fixed point structure theory.

Let $\mathscr{C}$ be a class of structured sets (ordered sets, ordered linear spaces, topological spaces, metric spaces, Hilbert spaces, Banach spaces, ...). Let Set^* be the class of nonempty sets and if X is a nonempty set, then

$$P(X) := \{Y \subset X \mid Y \neq \emptyset\}.$$

We shall use the following notations:

$P(\mathscr{C}) := \{U \in P(X) \mid X \in \mathscr{C}\}$,

$\mathbb{M}(U,V) := \{f : U \to V \mid f \text{ an operator}\}$,

$\mathbb{M}(U) := \mathbb{M}(U,U)$,

$S : \mathscr{C} \multimap Set^*,\ X \mapsto S(X) \subset P(X)$,

$M : D_M \subset P(\mathscr{C}) \times P(\mathscr{C}) \multimap \mathbb{M}(P(\mathscr{C}), P(\mathscr{C})),\ (U,V) \mapsto M(U,V) \subset \mathbb{M}(U,V)$.

We also consider the following multivalued operator:

$$P : \mathscr{C} \multimap Set^*,\ X \mapsto P(X).$$

Definition 0.1. *By a fixed point structure (f.p.s.) on $X \in \mathscr{C}$ we understand a triple $(X, S(X), M)$ with the following properties:*

(*i*) $U \in S(X) \Rightarrow (U,U) \in D_M$;

(*ii*) $U \in S(X),\ f \in M(U) \Rightarrow F_f := \{u \in U \mid f(u) = u\} \neq \emptyset$;

(*iii*) *M is such that:*

$$(Y,Y) \in D_M,\ Z \in P(Y),\ (Z,Z) \in D_M \Rightarrow M(Z) \supset \{f\big|_Z \mid f \in M(Y)\}.$$

A triple $(X, S(X), M)$ which satisfies (i) and (ii) is called a large fixed point structure (l.f.p.s.).

Example 0.1 (The *f.p.s.* of progressive operators)**.** *Let $\mathscr{C}$ be the class of partially ordered sets. For $(X, \leq) \in \mathscr{C}$, let*

$$S(X) := \{Y \in P(X) \mid (Y, \leq) \text{ has at least a maximal element}\}$$

and $M(Y) := \{f : Y \to Y \mid x \leq f(x),\ \forall\ x \in Y\}$. *We suppose that* $S(X) \neq \emptyset$ *and* $M(Y) \neq \emptyset$. *Then,* $(X, S(X), M)$ *is a f.p.s.*

Example 0.2 (The *f.p.s.* of contractions)**.** *$\mathscr{C}$ is the class of complete metric spaces,* $S(X) := P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$ *and* $M(Y) := \{f : Y \to Y \mid f \text{ is a contraction}\}$. *Then* $(X, S(X), M)$ *is a f.p.s.*

Example 0.3 (The *f.p.s.* of graphic contractions)**.** *$\mathscr{C}$ is the class of complete metric spaces,* $S(X) := P_{cl}(X)$ *and* $M(Y) := \{f : Y \to Y \mid d(f^2(x), f(x)) \leq \alpha d(x, f(x)),\ \forall\ x \in Y,\ 0 \leq \alpha < 1 \text{ and } f \text{ has a closed graphic}\}$. *Then* $(X, S(X), M)$ *is a f.p.s.*

Example 0.4 (The *f.p.s.* of Caristi)**.** *$\mathscr{C}$ is the class of complete metric spaces,* $S(X) := P_{cl}(X)$ *and* $M(Y) := \{f : Y \to Y \mid \exists\ \varphi : Y \to \mathbb{R}_+ \text{ l.s.c. such that } d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ \forall\ x \in Y\}$. *Then* $(X, S(X), M)$ *is a f.p.s.*

Example 0.5 (The *f.p.s.* of Schauder)**.** *$\mathscr{C}$ is the class of Banach spaces,* $S(X) := P_{cp,cv}(X) := \{Y \subset X \mid Y \text{ is compact and convex}\}$ *and* $M(Y) := C(Y, Y) := \{f : Y \to Y \mid f \text{ is continuous}\}$. *Then* $(X, S(X), M)$ *is a f.p.s.*

It is clear that for any fixed point theorem we have an example of *f.p.s.* or of a large *f.p.s.*

Definition 0.2. *If* $(X, S(X), M)$, $X \in \mathscr{C}$ *is a f.p.s. then an element* $Y \in S(X)$ *is called a fixed point space.*

The basic notions of the *f.p.s.* theory are the following:

Definition 0.3. *Let* X *be a nonempty set,* $Z \subset P(X)$, $Z \neq \emptyset$. *A functional* $\theta : Z \to \mathbb{R}_+$ *has the intersection property if* $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ *and* $\theta(Y_n) \to 0$ *as* $n \to \infty$ *imply that:*

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in Z \text{ and } \theta(Y_\infty) = 0.$$

Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$ and $\theta : Z \to \mathbb{R}_+$ be a functional.

Definition 0.4. *An operator* $f : X \to X$ *is a strong* (θ, φ)*-contraction if:*

(i) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a comparison function;*
(ii) $A \in Z \Rightarrow f(A) \in Z$;
(iii) $\theta(f(A)) \leq \varphi(\theta(A))$, $\forall\ A \in Z$.

If f *satisfies* $(i) + (ii)$ *and the condition*

(iii') $\theta(f(A)) \leq \varphi(\theta(A))$, $\forall\ A \in Z$ *with* $f(A) \subset A$,

then f *is called a* (θ, φ)*-contraction.*

Definition 0.5. *An operator* $f : X \to X$ *is strong* θ*-condensing if:*

(i) $A \in Z \Rightarrow f(A) \in Z$;
(ii) $A \in Z$, $\theta(A) \neq 0 \Rightarrow \theta(f(A)) < \theta(A)$.

If f *satisfies* (i) *and the condition*

(ii') $A \in Z,\ f(A) \subset A,\ \theta(A) \neq 0 \Rightarrow \theta(f(A)) < \theta(A)$,

then f is called θ-condensing operator.

Definition 0.6. *Let $(X, S(X), M)$, $X \in \mathscr{C}$ be a f.p.s., $\theta : Z \to \mathbb{R}_+$ with $S(X) \subset Z \subset P(X)$ and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$. By definition, the pair (θ, η) is compatible with $(X, S(X), M)$ if:*

- (i) *η is a closure operator such that $S(X) \subset \eta(Z) \subset Z$;*
- (ii) *$\theta(\eta(Y)) = \theta(Y)$, $\forall\ Y \in Z$;*
- (iii) *$F_\eta \cap Z_\theta \subset S(X)$, where $F_\eta := \{Y \subset X \mid \eta(Y) = Y\}$ and $Z_\theta := \{Y \subset Z \mid \theta(Y) = 0\}$.*

Remark 0.1. *For more considerations on the above notions see* [119]. *See also* [108], [110] *and* [111].

Remark 0.2. *For the basic fixed point theorems see* [156], [22], [36], [50], [138], [147], [79], [104], [107], [121], [2], [14]-[19], [26], [44], [46], [62]-[66], [70]-[74], [76], [85], [100], [165] ...

Remark 0.3. *For the set theory and the category theory see N. Bourbaki* [162], *S. Mac Lane* [163], *M. Barr and C. Wells* [6], *S. Mac Lane* [83], *G. Jameson* [67], *M.A. Khamsi and W.A. Kirk* [147], *F.W. Lawvere* [148], [154], *J. Lambek* [149], *A. Baranga* [150], *J. Soto-Andrade and F.J. Varela* [151], *M. Wand* [155], *J. Adámek, V. Koubek and J. Reiterman* [153], ...

Remark 0.4. *Let $\mathscr{C}$ be a concrete category, i.e., the objects of $\mathscr{C}$, ob $\mathscr{C}$, are structured sets. Let $S(\mathscr{C})$ be a nonempty class of objects of $\mathscr{C}$ and for each $A \in S(\mathscr{C})$, $M(A)$ be a nonempty subset of $Hom_{\mathscr{C}}(A, A)$. By definition the triple $(\mathscr{C}, S(ob\ \mathscr{C}), M)$ is a f.p.s. on $\mathscr{C}$ if each object $A \in S(ob\ \mathscr{C})$ has the fixed point property with respect to $M(A)$. For the fixed point theory in a category see F.W. Lawvere* [148], *A. Baranga* [150], *J. Soto-Andrade and F.J. Varela* [151], *M. Wand* [155], *J. Adámek, V. Koubek and J. Reiterman* [153], *I.A. Rus* [119] *(pp. 22-28), ...*

1. Problem of the invariant fixed point spaces

Let us have a fixed point theorem, T, and an operator f which does not satisfy the conditions of T. In which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f\big|_Y$, satisfies the condition of T ?

In the terms of *f.p.s.* this problem takes the following form:

Problem 1. Let $(X, S(X), M)$ be a *f.p.s.* on $X \in \mathscr{C}$ and $f : A \to A$ be an operator with $A \subset X$. In which conditions there exists $Y \subset A$ such that:

$$(a)\ Y \in S(X),\ (b)\ f(Y) \subset Y \text{ and } (c)\ f\big|_Y \in M(Y)\ ?$$

We have for this problem some concrete results and some abstract results. We begin our consideration on Problem 1 with some concrete results.

Example 1.1. *Problem 1 in the case of f.p.s. of progressive operators and f an increasing operator.*

Let $(X, S(X), M)$ be the f.p.s. of progressive operators, $U \subset X$ and $f : U \to U$ be an increasing operator. The problem is in which conditions there exists $V \subset U$ such that:

$$f(V) \subset V,\ V \in S(X) \text{ and } f\big|_V \in M(V)\ ?$$

Commentaries:

If we take $V := (LF)_f := \{x \in U \mid x \leq f(x)\}$, then $f(V) \subset V$ and $f\big|_V$ is progressive.

So, V is a solution of our problem if: $V \neq \emptyset$ and $Max(V, \leq) :=$ maximal element set of $(V, \leq) \neq \emptyset$.

Each of the following conditions:

(1) there exists $x_0 \in U$ such that $x_0 \leq f(x_0)$;
(2) $(U, \leq)$ has the least element;
(3) $(U, \leq)$ is a complete lattice;

implies that $V \neq \emptyset$.

Each of the following conditions:

(1) there exists $\sup V$;
(2) every chain in V has an upper bound;
(3) $(U, \leq)$ is a complete lattice;

implies that, $Max(V, \leq) \neq \emptyset$.

From the above considerations we have, for example, the following theorem of A. Tarski (1955):

If U is a complete lattice and $f : U \to U$ is increasing, then $F_f \neq \emptyset$.

The above considerations suggests us the following problem:

Problem 1.1. *Let X be a nonempty set and $f : X \to X$ be an operator. The problem is to construct an ordered relation, $\leq$, on X such that: $Max(X, \leq) \neq \emptyset$ and $f : (X, \leq) \to (X, \leq)$ is a progressive operator.*

As references for this problem see, for example, E. Bishop and R.R. Phelps [135], A. Brondsted [136], J. Jachymski [139], ...

For the fixed point theory in ordered sets see D. Duffus and I. Rival [40], H. Höft and M. Höft [57], B.S.W. Schröder [152], S. Heikkilä and V. Lakshmikantham [159], S. Carl and S. Heikkilä [160], I.A. Rus [107], I.A. Rus, A. Petruşel and G. Petruşel [121](pp. 11-19), A. Granas and J. Dugundji [50](pp. 25-28, 34-35) and the references therein.

Example 1.2. *Problem 1 in the case of f.p.s. of contractions and f an increasing operator.*

Let $(X, d, \leq)$ be an ordered metric space with d a complete metric, $S(X) := P_{cl}(X)$ and $M(Y) := \{f : Y \to Y \mid f$ is a contraction$\}$.

Let $U \subset X$ and $f : U \to U$ be an increasing operator. In which conditions there exists $V \subset U$ such that: $f(V) \subset V$, $V \in P_{cl}(X)$ and $f\big|_V : V \to V$ is a contraction ?

Commentaries:

In [9] the following result is given:

Blackwell's theorem (1965). *Let Ω be a nonemtpy set, $B(\Omega) := \big\{x : \Omega \to \mathbb{R} \mid x \text{ is bounded}\big\}$ and $\|x\| := \sup\limits_{t\in\Omega}|x(t)|$ and $\leq$ be the natural partial order relation on $B(\Omega)$. Let $E \subset B(\Omega)$ be a closed linear subspace containing the constant functions. Let $f : E \to E$ be an operator. We suppose that:*

- (i) *f is increasing;*
- (ii) *there exists $0 < \alpha < 1$ such that*
$$f(x + c) = f(x) + \alpha c$$
for all $x \in E$ and all constant function $c \in B(\Omega)$.

Then:

- (1) *f is an α-contraction;*
- (2) *$F_f = \{x^*\}$.*

In [50](p. 22) the authors put instead of (ii) the following condition:

(ii') there exists $0 < \alpha < 1$ such that $f(x + c) \leq f(x) + \alpha c$, for all $x \in E$ and all constant functions $c \in B(\Omega)$.

Let
$$e : \mathbb{R} \to B(\Omega),\ r \mapsto e(r) : \Omega \to \mathbb{R}$$
$$t \mapsto r$$
be an isometric embedding of $\mathbb{R}$ in $B(\Omega)$.

It is clear that: $\|e(r)\| = |r|$ and $x \leq y \Rightarrow x \leq e(\|y\|)$.

So, the condition (ii') takes the following form:

(ii') $f(x + e(r)) \leq f(x) + e(\alpha r),\ \forall\ x \in E,\ \forall\ r \in \mathbb{R}$.

By a similar proof as in [9] we have

Theorem 1.1. *We suppose that:*

- (i) *f is increasing;*
- (ii'') *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*
$$f(x + e(r)) \leq f(x) + e(\varphi(x)),\ \forall\ x \in E,\ \forall\ r \in \mathbb{R}_+.$$

Then:

- (1) *f is a φ-contraction;*
- (2) *$F_f = \{x^*\}$.*

From the above considerations the following problems arise

Problem 1.2. *To study the operators which satisfy: (a) condition (ii), (b) condition (ii'), (c) condition (ii'').*

Problem 1.3. *Let $(\mathbb{B}, +, \mathbb{R}, \|\cdot\|, K)$ be an ordered Banach space and $(E, +, \mathbb{R}, \|\cdot\|_K, \leq)$ be an ordered Banach space with a monotone K-norm, $\|\cdot\|_K : E \to K$. We suppose that there exists an isometric embedding $e : K \to E$ such that*
$$x, y \in E,\ x \leq y \Rightarrow x \leq e(\|y\|).$$

Let $f : E \to E$ be an increasing operator.

The problem is to study in which conditions the operator f satisfies the condition:

(ii''') *there exists a linear positive operator $S : \mathbb{B} \to \mathbb{B}$ with spectral radius less than 1 such that:*

$$f(x + e(k)) \leq f(x) + e(S_K), \ \forall\ x \in E, \ \forall\ k \in K.$$

For the linear K-normed spaces see: P.P. Zabrejko [140], I.A. Rus, A. Petruşel and G. Petruşel [121].

For the theory of embedding operators see: C. Bessaga and A. Pelczyński [8], A. Granas and J. Dugundji [50].

For the spectral radius of linear operators see: I. Gohberg, S. Goldberg and M.A. Kaashoek [157], J. Appell, E. De Pascale and A. Vignolli [5], P.P. Zabrejko [140], ...

Example 1.3. *Problem 1 in the case of f.p.s. of contractions and f a nonexpansive operator.*

Let $(X, S(X), M)$ be the f.p.s. of contractions, $U \subset X$ and $f : U \to U$ be a nonexpansive operator. In which conditions there exists $V \subset U$ such that: $f(V) \subset V$, $V \in P_{cl}(X)$ and $f\big|_V : V \to V$ is a contraction.

Commentaries:

Let us consider the Banach space $C[0,1]$ with sup norm. Let $f : C[0,1] \to C[0,1]$ be an operator. We suppose that f has an interpolation point $t_0 \in [0,1]$, i.e., $f(x)(t_0) = x(t_0)$, for all $x \in C[0,1]$. For $r \in \mathbb{R}$, let

$$X_r := \big\{x \in C[0,1] \mid x(t_0) = r, \ t_0 \in [0,1]\big\}.$$

Then, $f(X_r) \subset X_r$, $\forall\ r \in \mathbb{R}$.

The problem is to seek in which conditions $f\big|_{X_r} : X_r \to X_r$ is a contraction.

For example, for $n \in \mathbb{N}^*$, let $f := B_n$, where B_n is the Bernstein operator, i.e.,

$$B_n(x)(t) := \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} x\left(\frac{k}{n}\right).$$

It is well known that B_n is a nonexpansive operator and that $t = 0$ and $t = 1$ are interpolation points for B_n. So, for $\alpha, \beta \in \mathbb{R}$ the subset

$$X_{\alpha,\beta} := \big\{x \in C[0,1] \mid x(0) = \alpha, \ x(1) = \beta\big\}$$

is an invariant subset for B_n. Moreover

$$C[0,1] = \bigcup_{\alpha,\beta \in \mathbb{R}} X_{\alpha,\beta}$$

is a partition of $C[0,1]$, and

$$B_n\big|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$$

is a contraction for all $\alpha, \beta \in \mathbb{R}$ (see [117]).

For similar results in the case of nonexpansive linear increasing operators see: I.A. Rus [115], [118], [141], O. Agratini and I.A. Rus [142], Sz. Andras and I.A. Rus [137], J. Jachymski [64] and [65].

Example 1.4. *Let $(X, S(X), M)$ be a f.p.s, $Y \subset X$ and $f \in M(Y)$ be an involution of order n (i.e., $f^n = 1_Y$). If there exists $Z \subset Y$ such that $U := Z \cup f(Z) \cup \ldots \cup f^{n-1}(Z) \in S(X)$, then $U \in I(f)$ and $f\big|_U \in M(U)$.*

References: [113], [164].

For some abstract results for Problem 1 see: [119](pp. 69-89). See also [108], [110], [113] and [121]. For example we have:

Theorem 1.2 ([119]). *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathbb{R}_+)$ be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(*i*) *$\theta\big|_{\eta(Z)}$ has the intersection property;*
(*ii*) *f is a (θ, φ)-contraction.*

Then:

(*a*) *there exists $U \in S(X)$ such that $U \subset Y$ and $f(U) \subset U$;*
(*b*) *$F_f \neq \emptyset$;*
(*c*) *if $F_f \in Z$, then $\theta(F_f) = 0$.*

Theorem 1.3. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathbb{R}_+)$ be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(*i*) *$A \in Z$, $x \in Y$ imply that $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;*
(*ii*) *f is a θ-condensing operator.*

Then:

(*a*) *there exists $U \in S(X)$ such that $U \subset Y$ and $f(U) \subset U$;*
(*b*) *$F_f \neq \emptyset$;*
(*c*) *if $F_f \in Z$ then $\theta(F_f) = 0$*

These general results generalize some results given by: G. Darbo (1955), B.N. Sadovskij (1967), R.D. Nussbaum (1969), M. Furi and A. Vignoli (1970), J. Danes (1976), F.S. De Blasi (1977), L. Pasiki (1979), G.S. Jones (1973), W.V. Petryshyn (1973), S. Reich (1971), J. Eisenfeld and V. Lakshmikantham (1975), J.K. Hale and O. Lopes (1973), ... See: [36], [50], [94], [97], [138], [145], [119], ...

For the invariant-subspace problem see [30] and the references therein.

2. Problem of the maximal fixed point structures

Let $(X, S(X), M)$ be a *f.p.s.* and $S_1(X) \subset P(X)$ with $S_1(X) \supset S(X)$.

Problem 2. *Which are the f.p.s. with the following property:*

$$S(X) = \{A \in S_1(X) \mid f \in M(A) \Rightarrow F_f \neq \emptyset\} \ ?$$

By definition, a solution of this problem is called a maximal *f.p.s.* in $S_1(X)$. If $S_1(X) = P(X)$, then it is called a maximal *f.p.s.*

Problem $2_\mathbf{a}$. *Let $(X, S(X), M)$ be a maximal f.p.s. in $S_1(X)$. Does there exists $S_2(X) \supset S_1(X)$ such that $(X, S(X), M)$ is maximal in $S_2(X)$?*

Problem $2_\mathbf{b}$. *Let $(X, S(X), M)$ be a maximal f.p.s. in $S_1(X)$. Does there exists $M_1 \subset M$ such that $(X, S(X), M_1)$ is maximal in $S_1(X)$?*

Commentaries:

Example 2.1 (A.C. Davis (1955; see [107], [108], [121]). *$(X, \leq)$ is a complete lattice, $S(X) := \{Y \in P(X) \mid (Y, \leq)$ is a complete lattice$\}$ and $M(Y) := \{f : Y \to Y \mid f$ is increasing$\}$, $S_1(X) := \{Y \in P(X) \mid (Y, \leq)$ is a lattice$\}$. Then $(X, S(X), M)$ is maximal in $S_1(X)$.*

Example 2.2 (E.H. Conell (1959), P.V. Subrahmanyan (1975); see [107], [121]). *The f.p.s. of contractions is not maximal.*

Example 2.3 (P.V. Subrahmanyan (1975)). *The f.p.s. of Kannan in maximal.*

Example 2.4 (W.A. Kirk (1976; [72])). *The f.p.s. of Caristi is maximal.*

Example 2.5 (S. Park (1984; [89])). *The f.p.s. of graphic contractions is maximal.*

Example 2.6 (V. Klee (1955; [77])). *The f.p.s of Schauder is maximal in $S_1(X) := P_{b,cl}(X)$.*

Example 2.7 (P.K. Lin and Y. Sternfeld (1985; [81]). *The f.p.s. of Schauder is maximal in $S_1(X) := P_{b,cv}(X)$ and $M_1(A) = Lip(A, A) := \{f : A \to A \mid f$ is a Lipschitz operator$\}$.*

Example 2.8 (M.C. Anisiu and V. Anisiu (1997; [4])). *Let $\mathbb{B}$ be a Banach space, $A \subset \mathbb{B}$ be a convex set with $intA \neq \emptyset$. If each contraction $f : A \to A$ has a fixed point, then A is closed.*

For other results in this directions see: [1], [4], [72], [82], [89], [107], [116], [119], [120], [130], [133], [139], ...

For some general considerations of maximal *f.p.s.* see [119](pp. 32-34). See also [120].

3. Problem of the pairs of operators, in a fixed point structure, with common fixed points

Let us have a fixed point theorem and, f and g be two operators which satisfy the conditions of this theorem. In which conditions we have that $F_f \cap F_g \neq \emptyset$?

Which are the theorems T with the following property:

If f and g satisfy the conditions of T and $f \circ g = g \circ f$, then $F_f \cap F_g \neq \emptyset$?

In the terms of *f.p.s.* these problems take the following form:

Problem 3. *Let $(X, S(X), M)$ be a f.p.s., $Y \in S(X)$ and $f, g \in M(Y)$. In which conditions we have that $F_f \cap F_g \neq \emptyset$?*

Problem $\mathbf{3_a}$. *Which are the f.p.s., $(X, S(X), M)$ with the following property:*

$$Y \in S(X),\ f, g \in M(Y),\ f \circ g = g \circ f \Rightarrow F_f \cap F_g \neq \emptyset \text{ ?}$$

By definition, a solution of this problem is called a *f.p.s.* with the common fixed point property.

Commentaries:

Example 3.1. *For the Brouwer's f.p.s. in the case $n = 1$, examples and counterexamples are given by: H. Cahen (1964), J.H. Folkman (1966), J.P. Huneke (1969), W.M. Boyce (1969), J.P. Huneke and H.H. Glover (1971), J.R. Jachymski (1996), W.A. Kirk (2010), ... See:* [49], [62], [113], [114], [119], [121], ... *The Brouwer's f.p.s. is not with the common fixed point property.*

Example 3.2. *For the Browder-Ghöde-Kirk's f.p.s. some results are given by: R.E. Bruck (1974), T. Suzuki (2004), K. Goebel (2010), ... See:* [25], [33], [127], [138], [50], [119], [121], ...

Example 3.3. *The following result is well known:*

Brunel's theorem (1970; see [37]). *Let X be a Banach space, $f, g : X \to X$ be two linear nonexpansive operators with $f \circ g = g \circ f$ and $\lambda \in]0, 1[$. Then each fixed point of $\lambda f + (1 - \lambda)g$ is a common fixed point of f and g.*

This result suggest us the following question:

Problem 3.1. *Let X be a Banach space and $f, g : X \to X$ be two linear nonexpansive operators with $f \circ g = g \circ f$. Let $G : X \times X \to X$ be such that (see* [166]*):*

(A_1) $G(x, x) = x, \ \forall\ x \in X$;
(A_2) $x, y \in X,\ G(x, y) = x \ \Rightarrow\ y = x$;
(A_3) *G is a linear operator.*

Let $T : X \to X$ be the operator defined by $T(x) := G(f(x), g(x))$, for all $x \in X$. The problem is in which conditions each fixed point of T is a common fixed point of f and g ?

For the fixed point structure with the common fixed point property see [114] and [119].

Now we present two abstract results (see [119], pp. 92-97).

Theorem 3.1. *Let $(X, S(X), M)$ be a f.p.s. with the common f.p.p. and (θ, η) ($\theta : Z \to \mathbb{R}_+$) be a compatible pair with $(X, S(X), M)$. We suppose that $Y \in \eta(Z)$, $f, g \in M(Y)$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

(i) *$\theta\big|_{\eta(Z)}$ has the intersection property;*
(ii) $f \circ g = g \circ f$;
(iii) *$\theta(f(A) \cup g(A)) \leq \varphi(\theta(A))$, $\forall\ A \in Z$ such that*

$$f(A) \subset A \text{ and } g(A) \subset A.$$

Then:

(a) $F_f \cap F_g \neq \emptyset$;
(b) *if $F_f \cap F_g \in Z$, then $\theta(F_f \cap F_g) = 0$.*

If in the Theorem 3.1 we take the following fixed point structure on a strictly convex Banach space X, $S(X) := P_{cp,cv}(X)$ and $M(Y) := \big\{h : Y \to Y \mid h \text{ is a nonexpansive operator}\big\}$ and $\theta := \alpha_K$, the Kuratowski measure of noncompactness, then we have

Theorem 3.2. *Let X be a strictly convex Banach space, $Y \in P_{b,cl,cv}(X)$ and $f, g : Y \to Y$ be two nonexpansive operators. We suppose that:*

(*i*) $f \circ g = g \circ f$;
(*ii*) *the pair* (f, g) *is an* (α_k, φ)*-contraction pair.*

Then:

(*a*) $F_f \cap F_g \neq \emptyset$;
(*b*) $F_f \cap F_g$ *is a compact subset.*

For the common fixed point theory see: [50], [121], [11], [33], [36], [37], [39], [49], [52], [62], [74], [107], [114], [119], [128] ,[132], [138], ...

4. Problem of the pairs of operators, in a fixed point structure, with coincidence points

Let T be a fixed point theorem and, f and g be two operators which satisfy the conditions of T.

In which conditions we have that, $C(f, g) \neq \emptyset$?

Which are the theorems T with the following property:

If f *and* g *satisfy the condition of* T *and* $f \circ g = g \circ f$, *then* $C(f, g) \neq \emptyset$ *?*

In the terms of *f.p.s.* these problems take the following form:

Problem 4. *Let* $(X, S(X), M)$ *be a f.p.s.,* $Y \in S(X)$ *and* $f, g \in M(Y)$*. In which conditions we have that,* $C(f, g) \neq \emptyset$ *?*

Problem 4_a. *Which are the f.p.s.,* $(X, S(X), M)$ *with the following property:*

$$Y \in S(X),\ f, g \in M(Y),\ f \circ g = g \circ f \Rightarrow C(f, g) \neq \emptyset \ ?$$

By definition, a solution of Problem 4_a is called a *f.p.s.* with the coincidence property.

Commentaries:

Example 4.1 (W.A. Horn (1970; [144])**.** *The f.p.s. of Brouwer in the case* $n = 1$ *is a f.p.s. with the coincidence property.*

Problem 4_a in the case of the *f.p.s.* of Schauder is:

Horn's conjecture. *Let* X *be a Banach space,* $Y \in P_{cp,cv}(X)$, $f, g \in C(Y, Y)$*. If,* $f \circ g = g \circ f$*, the* $C(f, g) \neq \emptyset$.

This conjecture includes:

Schauder-Browder-Nussbaum's conjecture. *Let* X *be a Banach space,* $Y \in P_{b,cl,cv}(X)$ *and* $f : Y \to Y$ *be an operator. We suppose that:*

(*i*) $f \in C(Y, Y)$;
(*ii*) *there exists* $n_0 \in \mathbb{N}^*$ *such that,* f^{n_0} *is compact.*

Then, $F_f \neq \emptyset$.

For the above conjectures see: F.E. Browder [18], R.D. Nussbaum [145], W.A. Horn [144], R. Sine [143], I.A. Rus [112], V. Šeda [122], ...

Another aspect of Problem 4 is defined by:

Problem 4_b. *Which are the f.p.s.,* $(X, S(X), M)$*, with the following property:*

For each $Y \in S(X)$ *there exists* $p_Y : Y \to Y$ *such that:*

$$f \in M(Y, Y) \Rightarrow C(f, p_Y) \neq \emptyset \ ?$$

By definition such an operator, p_Y, is called coincidence producing operator in $(X, S(X), M)$.

For the Problem 4_b see I.A. Rus [146] and the references therein.

For some abstract results for Problem 4 see [119]. See also [113]. For example we have (see [119], pp. 102-107):

Theorem 4.1. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ be a compatible pair with $(X, S(X), M)$. Let $U \in \eta(Z)$ and $f, g : U \to U$ be two operators. We suppose that:*

(i) $A \in Z \Rightarrow P(A) \subset Z$;
(ii) $\theta\big|_{\eta(Z)}$ has the intersection property;
(iii) g has a left-inverse, g_l^{-1}, $f(U) \subset g(U)$ and $g_l^{-1} \circ f \in M(U)$;
(iv) there exist $\alpha, \beta \in \mathbb{R}_+^*$, $\alpha\beta < 1$, such that:
  (a) $\alpha\theta(g(A)) \geq \theta(A)$, for all $A \in P(U)$ with $f(A) \subset g(A)$;
  (b) $\theta(f(A)) \leq \beta\theta(A)$, for all $A \in P(U)$ with $f(A) \subset g(A)$.

Then, $C(f, g) \neq \emptyset$ and $\theta(C(f, g)) = 0$.

Theorem 4.2. *Let $(X, S(X), M)$ be a f.p.s. with the coincidence property, (θ, η) be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f, g \in M(Y)$ such that $f \circ g = g \circ f$.*

We suppose that the pair (f, g) is a θ-condensing pair.

Then, $C(f, g) \neq \emptyset$.

For the coincidence point theory see: [161], [6], [7], [156], [36], [41], [50], [60], [83], [85], [121], . . .

5. Problem on the nonself operators defined on a fixed point space

Problem 5. *Let $(X, S(X), M)$ be a f.p.s., $Y \in S(X)$ and $f \in M(Y, X)$. In which conditions we have that, $F_f \neq \emptyset$?*

Example 5.1 (The case of *f.p.s.* of contractions)**.** *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a contraction. In which conditions we have that, $F_f \neq \emptyset$?*

Example 5.2 (The case of *f.p.s.* of Schauder)**.** *Let X be a Banach space, $Y \in P_{cp,cv}(X)$ and $f \in C(Y, X)$. In which conditions we have that, $F_f \neq \emptyset$?*

A technique to solve our problem is suggested by the following problem.

Problem 5_a. *Let $(X, S(X), M)$ be a f.p.s., $Y \in S(X)$. For a given $f \in M(Y, X)$ find an operator $\rho_f : Y \to Y$ such that:*

$$(a)\ \rho_f \in M(Y) \text{ and } (b)\ F_f = F_{\rho_f};$$

or

$$(a)\ F_{\rho_f} \neq \emptyset \text{ and } (b)\ F_f = F_{\rho_f}.$$

We call such a ρ_f a generalized retract of f.

Example 5.3 (The case of *f.p.s.* of contractions)**.** *Let (X,d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be an α-contraction.*

Following Caristi [28], *we suppose that f is metrically inward, i.e., for each $x \in Y$, there exists $y \in Y$ such that $d(x,y)+d(y,f(x)) = d(x,f(x))$, where $y = x$ iff $x = f(x)$. In this condition we define the multivalued operator*

$$R_f : Y \multimap Y,\ R_f(x) := \begin{cases}]x, f(x)]_d \cap Y, & for\ x \in Y \setminus F_f \\ \{x\}, & for\ x \in F_f \end{cases}$$

where $]x,f(x)]_d$ is the metrical interval corresponding to d. Let ρ_f be a selection of R_f. It is clear that $F_{R_f} = F_{\rho_f} = F_f$. The problem is to prove that $F_{\rho_f} \neq \emptyset$. To do this, we remark that

$$d(x, \rho_f(x)) \leq \varphi(x) - \varphi(\rho_f(x)),\ \forall\ x \in Y,$$

where $\varphi(x) = (1-\alpha)^{-1} d(x,f(x))$.

From the Caristi's fixed point theorem it follows that $F_{\rho_f} \neq \emptyset$.

From the above considerations we have

Caristi's Theorem. *Let (X,d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$. We suppose that:*

(i) *f is a contraction;*

(ii) *f is metrically inward.*

Then, $F_f = \{x^*\}$.

The following problem suggests a "concrete" way to find a generalized retract of a given operator.

Problem 5_b. *Which are the f.p.s., $(X, S(X), M)$ with the following property:*

For each $Y \in S(X)$ there exists a set retraction $\rho : X \to Y$ such that for all $f \in M(Y,X)$, $\rho \circ f \in M(Y)$ or $F_{\rho \circ f} \neq \emptyset$.

We call an f for which $F_f = F_{\rho \circ f}$, retractible with respect to ρ and $\rho \circ f$ a retract of f (see R.F. Brown [21]).

The problem is to find which boundary conditions, which inwardness conditions and which outwardness conditions imply that $F_f = F_{\rho \circ f}$?

Problem 5_c (Conjecture of the generalized retracts)**.** *Each boundary condition (inwardness, outwardness) on f implies the existence of a generalized retract of the non-self operator f.*

Commentaries:

For the retract theory see: K. Borsuk [12], S. Hu [59], R.F. Brown [21], [20], M.C. Anisiu [3], F.E. Browder [14], F.E. Browder and W.V. Petryshyn [19], R.E. Bruck [23], [24], [25], F. Deutsch [38], D. Duffus and I. Rival [40], R. Espinola and A. Fernández-León [42], R. Espinola and G. López [43], D. Grundmane [51], A. Horvat-Marc [58], E.M. Jawhari, D. Misane and M. Pouzet [68], W.A. Kirk [73], E. Kopecká and S. Reich [78], J.J. Moreau [87], W.V. Petryshyn [91]-[94], A.J.B. Potter [95], I.A. Rus [109]-[119], I. Singer [123], R.D. Nussbaum [145], G. Isac [60], G. Isac and A.B. Németh [61].

For boundary (inwardness, outwardness, ...) conditions see: W.A. Kirk and C.H. Morales [75], J.A. Gatica and W.A. Kirk [47], A. Carbone and S.P. Singh [27], J. Caristi [28], J. Caristi and W.A. Kirk [29], D.G. De Figueiredo [34], [35], K. Fan [44], M. Fečkan [45], M. Frigon [46], B.R. Halpern [53], B.R. Halpern and G.M. Bergman [54], H. Brezis [13], A. Jiménez-Melado and C.H. Morales [69], L. Pasiki [90], W.V. Petryshyn [93], S. Reich [101], [102], J. Reinermann [104], J. Reinermann and R. Schöneberg [105], D. Roux and S.P. Singh [106], I.A. Rus [119], T.E. Williamson [131], ...

Other type of condition which appears in the fixed point theory for nonself operators we find in:

Problem 5_d. *Let $(X, S(X), M)$ be a f.p.s., $Y \in S(X)$ and $f : Y \to X$ such that $Y \subset f(Y)$. In which conditions we have that, $F_f \neq \emptyset$?*

Commentaries:

If $f_r^{-1} : f(Y) \to Y$ is a right inverse of f then $f_d^{-1}(Y) \subset Y$. So, if $f_d^{-1} \in M(Y)$, then, $F_f \neq \emptyset$.

References: O.H. Hamilton [55], J. Andres [2], T.L. Hicks and L.M. Saliga [56], J. Andres, K. Pastor and P. Snyrychova [158], ...

For others problems in the fixed point theory for nonself operators see: K. Deimling [36], A. Granas and J. Dugundji [50], M.A. Krasnoselskii and P. Zabrejko [79], D. O'Regan and R. Precup [88], I.A. Rus [107], [119], I.A. Rus, A. Petruşel and G. Petruşel [121], W.A. Kirk and B. Sims [138], V. Berinde [7], F.E. Browder [15], [16], [17], R.F. Brown [20], [21], A. Chiş-Novac [31], A. Chiş-Novac, R. Precup and I.A. Rus [32], R. Precup [96]-[99], D. Reem, S. Reich and J. Zaslavski [100], S.P. Singh, B. Watson and P. Srivastava [124], L.E. Ward [129], H.-K. Xu [134], ...

As an exotic result we mention the following one, given by D. Reem, S. Reich and A.J. Zaslavski (see [100]):

Let (X, d) be a complete metric space $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be a contraction. We suppose that there exists a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in Y such that $f^i(x_n)$ is defined for $i = \overline{1, n}$. Then f has a unique fixed point.

The problem is which classes of generalized contractions have the above property ?

As an abstract result for Problem 5 we present the following (see [119], pp.111-112)

Theorem 5.1. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$, $f : Y \to X$ be an operator and $\rho : X \to Y$ be a set retraction. We suppose that:*

(i) $\theta\big|_{\eta(Z)}$ is a functional with intersection property;
(ii) f is retractible with respect to ρ and $\rho \circ f \in M(Y)$;
(iii) ρ is (θ, l)-Lipschitz $(l \in \mathbb{R}_+)$;
(iv) f is a strong (θ, φ)-contraction;
(v) the function $l\varphi$ is a comparison function;

Then $F_f \neq \emptyset$ and if $F_f \in Z$, then $\theta(F_f) = 0$.

For other abstract results see [119], pp. 111-119. See also, [58], [109], [121], ...

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