Image of circles through remarkable analytical functions

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ABSTRACT

- Geometric Function Theory
  = Complex Analysis + Geometry;
- Applied Computer Science in Mathematics;
- Remarkable Analytical Functions Analysis;
- Special Classes of Univalent Functions;
- Special properties seen in function’s graphical representation.
SPECIAL CLASSES OF UNIVALENT FUNCTIONS

- The class of univalent functions in the unit disc and normalized;
- The class of starlike functions;
- The class of convex functions;
- The class of $\alpha$-convex functions;
- The class of spiralike functions.
IMAGE OF CIRCLES THROUGH REMARKABLE ANALYTICAL FUNCTIONS

- Observing special properties in function’s graphical representation;
- Using Maple for representing these images.
UNIVALENT FUNCTIONS

Definition
An holomorphic function and injective on the domain $D$ from $\mathbb{C}$ is univalent on $D$.

Remark
We denote by $\mathcal{H}_u(D)$ the set of univalent functions on the domain $D$. 
Univalent functions

Application

Given \( f_a(z) = z + az^2, a \in \{0.3, 0.5, 0.6\}, r \in \{0.5, 0.7, 0.9, 1\}, \) find the image of circles \( C_r = C(O; r), \) where \( r \) is the radius and \( O \) the center of the circle and the origin of the cartesian system.
UNIVALENT FUNCTIONS

Figure: $a=0.3, r=0.5$

Figure: $a=0.3, r=0.7$

Figure: $a=0.3, r=0.9$
UNIVALENT FUNCTIONS

Figure: $a=0.5, r=0.5$

Figure: $a=0.5, r=0.7$

Figure: $a=0.5, r=0.9$
UNIVALENT FUNCTIONS

Figure: $a=0.6, r=0.5$

Figure: $a=0.6, r=0.7$

Figure: $a=0.6, r=0.9$
**S CLASS**

**THE CLASS OF UNIVALENT FUNCTIONS IN THE UNIT DISC AND NORMALIZED**

**Definition**

\[ S = \{ f \in \mathcal{H}_u(U) \mid f(0) = 0, f'(0) = 1 \} \]

**Example**

*Function f : \mathbb{C} \to \mathbb{C}, f(z) = az^2 + z with a \in (-\frac{1}{2}, \frac{1}{2}) is univalent.*
**S* CLASS**

**THE STARLIKE FUNCTIONS**

**Definition**

\[ S^* = \left\{ f \in \mathcal{H}(U) \mid f(z) = z + a_2z^2 + \ldots, \quad Re\left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in U \right\}. \]
**S* class**

**The starlike functions**

Application

*Given the Koebe function* \( k(z) = \frac{z}{(1 - z)^2} \),

\( r \in \{0.5, 0.6, 0.7, 0.8, 0.9\} \) *find the image of circles* \( C_r = C(O; r) \).
S* CLASS
THE STARLIKE FUNCTIONS

Figure: r=0.5
Figure: r=0.6
Figure: r=0.7
CLASA $S^*$
THE STARLIKE FUNCTIONS

Figure: $r=0.8$

Figure: $r=0.9$
**Clasa $S^*$**

**The starlike functions**

**Application**

Function $f(z) = -2z + z^2 + 6 \log \frac{2 + z}{z}$ is a function with a positive real part of the derivative. This function is not a starlike function.
**CLASA S***

**THE STARLIKE FUNCTIONS**

Figure: \( r=1 \)

Figure: \( r=1, \text{zoom} \)
**K CLASS**

**THE CLASS OF CONVEX FUNCTIONS**

**Definition**

\[ \mathcal{K} = \{ f \in A : \text{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \} \]
**K CLASS**

**THE CLASS OF CONVEX FUNCTIONS**

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**Application**

*Given the Koebe convex function*  
\[ s(z) = \frac{z}{1 - z}, \]

*where*  
\[ r \in \{0.5, 0.6, 0.7, 0.8, 0.9\} \text{ find the image of circles } C_r = C(0; r). \]

*The function is called the Koebe convex function because*  
\[ zs'(z) = k(z). \]
K CLASS
THE CLASS OF CONVEX FUNCTIONS

Figure: $r=0.5$

Figure: $r=0.6$

Figure: $r=0.7$
K class
The class of convex functions

Figure: $r=0.8$

Figure: $r=0.9$
THE CLASS OF SPIRALIKE FUNCTIONS

Definition

A function $f$ is a $\gamma$-spiralike function in $U$ if $f \in \mathcal{H}_u(U)$, with $f(0) = 0$, and $U$ is a $\gamma$-spiralike domain.
THE CLASS OF SPIRALIKE FUNCTIONS

Application

Given $f(z) = \frac{z}{(1 - z)^2 e^{-i\gamma} \cos \gamma}$, find the image of circles $C_r = C(0; r)$ for different values of $\gamma$ and $r$. 
THE CLASS OF SPIRALIKE FUNCTIONS

Figure: $\gamma = \frac{\pi}{6}, r=0.4$

Figure: $\gamma = \frac{\pi}{6}, r=0.6$

Figure: $\gamma = \frac{\pi}{6}, r=0.8$
The class of spiralike functions

Figure: \( \gamma = \frac{\pi}{4}, r=0.4 \)

Figure: \( \gamma = \frac{\pi}{4}, r=0.6 \)

Figure: \( \gamma = \frac{\pi}{4}, r=0.8 \)
THE CLASS OF SPIRALIKE FUNCTIONS

Figure: $\gamma = \frac{\pi}{3}, r=0.4$

Figure: $\gamma = \frac{\pi}{3}, r=0.6$

Figure: $\gamma = \frac{\pi}{3}, r=0.8$
Application

Given the functions \( f(z) = z + az^2 \) with negative coefficients, find the image of the unit disc for \( a \in \left\{ \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4} \right\} \)
Extremal functions

- $f_2(z) = z - \frac{z^2}{2}$ is an extremal function for the class of starlike functions;
- for $n = 2$ the image of the unit disc through this function is a starlike set;
- $f_4(z) = z - \frac{z^2}{4}$ is an extremal function for the class of convex functions;
- pentru $n = 4$ the image of the unit disc through this function is a convex set;
- $f_3(z) = z - \frac{z^2}{3}$ is not an extremal function.
EXTREMAL FUNCTIONS

Figure: r=1, a=-1/2

Figure: r=1, a=-1/3

Figure: r=1, a=-1/4
**Open-door function**

*Application*

Function \( f(z) = \frac{1 + z}{1 - z} + \frac{2n\alpha z}{1 - z^2} \) is called the open-door function. Given that, find the image of circles through this function.

The open-door function was named by S. S. Miller and P. T. Mocanu, because through it, the unity disc is transformed into the complex plane cut by a little door, formed by: \( \text{Re}\omega = 0 \) and \( |\text{Im}\omega| \geq \sqrt{n\alpha(n\alpha + 2)} \), so left-half and right-half planes are united through an open door.
**Open-door function**

**Figure:**

\[ r = 0.8, \alpha = 1, n = 3 \]

**Figure:**

\[ r = 0.8, \alpha = 1, n = 1 \]
Open-door function

Figure: $r=0.8, \alpha = 1, n=2$

Figure: $r=0.8, \alpha = 100, n=1$
Thank you!

Q & A