REFINEMENTS OF THE GERRETSEN INEQUALITY

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Abstract. Gerretsen inequality states that in a triangle the sum of squares of the sides is less or equal than a linear combination of the squares of R and r, where R is the radius of the circumscribed circle and r is the radius of the inscribed circle. The purpose of this article is to find a chain of inequalities which represent refinements of the Gerretsen inequality. In the chain of the inequalities are included rational functions of R and r and the square root of a fourth degree polynomial in R and r.

1. Introduction

Let ABC be a triangle. We shall denote $a = BC$, $b = AC$, $c = AB$, $s = \dfrac{a+b+c}{2}$ the semiperimeter, R the radius of the circumscribed circle and r the radius of the inscribed circle.

In [6] J.C. Gerretsen proved that in any triangle ABC the following inequality $a^2 + b^2 + c^2 \le 8R^2 + 4r^2$ holds.

In the paper [11] was studied the following problem: *Find the best constants $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \ge 0$ such that the following inequality holds*

$$a^2 + b^2 + c^2 \le \alpha R^2 + \beta Rr + \gamma r^2.$$

In [10] L. Panaitopol proved that if $\beta = 0$ then $\alpha = 8$ and $\gamma = 4$ are the best constants for which the above inequality holds.

In [11] was proved that the constants $\alpha = 8$, $\beta = 0$ and $\gamma = 4$ are the best constants for which the above inequality holds. In other words it was proved that the constants in the inequality of Gerretesen are the best constants.

This means that if α, β and γ are real numbers with $\beta \ge 0$ and with the property that the inequality $a^2+b^2+c^2 \le \alpha R^2+\beta Rr+\gamma r^2$ is true in every triangle ABC, then we have that the inequality $8R^2+4r^2 \le \alpha R^2+\beta Rr+\gamma r^2$ is true in every triangle ABC.

In [4] W.J. Blundon proposed the following inequality

$$|s^2 - (2R^2 + 10Rr - r^2)| \le 2(R - 2r)\sqrt{R(R - 2r)}\,.$$

In [5] W.J. Blundon gave a proof of the above inequality. In the following we shall refer to the preceding inequality as the Blundon inequality.

Also in [7] A. Makovski gives another solution to Blundon's inequality.

In [11] was proved the inequality

$$a^2 + b^2 + c^2 \le \frac{36(8R^4 + tr^4)}{36R^2 + (t - 16)r^2}\,, \text{ for each } t \in [-2, 6]. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 26D15, 51M16.

Key words and phrases. Triangle inequalities, Gerretsen inequality, inscribed, circumscribed.

If we take in (1.1) $t = 6$, we obtain the inequality

$$a^2 + b^2 + c^2 \le \frac{36(4R^4 + 3r^4)}{18R^2 - 5r^2}. \qquad (1.2)$$

Inequality

$$a^2 + b^2 + c^2 \le \frac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 - 8r^2} \qquad (1.3)$$

was proved in [1] by A. Bager. It represents an improvement of the inequality

$$a^2 + b^2 + c^2 \le \frac{(4R + r)(4R^2 - 3Rr + 2r^2)}{2R - r}. \qquad (1.4)$$

One can easily to prove that (1.2) represent an improvement of (1.3) (see Theorem 2.4). Also inequality (1.4) represent an improvement of the inequality $a^2 + b^2 + c^2 \le \frac{72R^4}{9R^2 - 4r^2}$ proposed by I.V. Maftei to "Arhimede International Symposium of Pure and Applied Mathematics 2008" and solve in [2] by D. Băiţan.

Consider the function $f : [-2, +\infty) \to \mathbb{R}$

$$f(t) = \frac{36(8R^4 + tr^4)}{36R^2 + (t - 16)r^2}. \qquad (1.5)$$

In [11] was proved that f is decreasing. It results that $f(6) \le f(1) \le f(0) \le f(-1) \le f(-2)$.
The above chain of inequalities may be written as follows

$$a^2 + b^2 + c^2 \le \frac{36(4R^4 + 3r^4)}{18R^2 - 5r^2} \le \frac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 - 8r^2}$$
$$\le \frac{(4R + r)(4R^2 - 3Rr + 2r^2)}{2R - r} \le \frac{72R^4}{9R^2 - 4r^2} \le 8R^2 + 4r^2,$$

which prove that the inequality (1.2) is the best from this chain.
Note that the above chain of inequalities contains inequalities (1.2), (1.3) and (1.4).

2. Main results

In the following we will determine the constants $\alpha, \beta, \gamma, \Delta$ such that the inequality

$$a^2 + b^2 + c^2 \le \frac{\alpha R^4 + \beta r^4}{\gamma R^2 + \Delta r^2} \le 8R^2 + 4r^2 \qquad (2.1)$$

to be true in any ABC triangle and the inequality

$$a^2 + b^2 + c^2 \le \frac{\alpha R^4 + \beta r^4}{\gamma R^2 + \Delta r^2}$$

to be the best of this type. We denote $u = \frac{\alpha}{\gamma}$, $v = \frac{\Delta}{\gamma}$, $t = \frac{\beta}{\gamma}$. Inequality (1.5) may be written as:

$$a^2 + b^2 + c^2 \le \frac{uR^4 + tr^4}{R^2 + vr^2} \le 8R^2 + 4r^2. \qquad (2.2)$$

If we consider the case of isosceles triangle with the sides $a = 0, b = c = 1$ we have that $R = \frac{1}{2}$, $r = 0$. We replace in (2.2) and obtain $u = 8$.

If we consider the case of equilateral triangle, from (2.2) we obtain that $v = \dfrac{t-16}{36}$.

The inequality (2.2) may be written equivalent as

$$a^2+b^2+c^2 \le \frac{36\left(8R^4+tr^4\right)}{36R^2+(t-16)\,r^2} \le 8R^2+4r^2. \tag{2.3}$$

The right side of inequality (2.3) after we replace with $x=\dfrac{R}{r}$ may be written as

$$\frac{72x^4+9t}{32x^2+t-16} \le 2x^2+1$$

and after we perform some calculation we obtain

$$\frac{(2t+4)\left(-x^2+4\right)}{36x^2+t-16} \le 0 \text{ for each } x \ge 2. \tag{2.4}$$

If we take in (2.4) $x\to\infty$ we obtain $t\ge -2$.

In [11] was proved that the function f defined by (1.5) is decreasing and $a^2+b^2+c^2 \le f(t) \le 8R^2+4r^2$ for every $t\in[-2,6]$.
We propose to find the greatest $t_0\in(6,\infty)$ for which the above inequality holds in any triangle.

Theorem 2.1. *In any ABC triangle the following chain of inequalities is true:*

$$a^2+b^2+c^2 \le \frac{36\left(8R^4+tr^4\right)}{36R^2+(t-16)\,r^2} \le 8R^2+4r^2, \text{ for each } t\in[-2,t_0] \tag{2.5}$$

where x_0 is the unique positive root of the equation:

$$18x^4-82x^3-4x^2-3x-2=0,$$

$x_0\approx 4,61269$ *and F is the function $F:(2,+\infty)\to R$,*

$$F(x)=\frac{72x^4+72x^3-216x^2-60x+40-(72x^3+144x^2-36x-72)\sqrt{x^2-2x}}{12x+25}$$

and $t_0=F(x_0)\approx 6.06944$.

Proof. From the equality

$$a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right)$$

the left side of the inequality (2.5) may be written as

$$s^2 \le r^2+4Rr+\frac{18\left(8R^4+tr^4\right)}{36R^2-(16-t)\,r^2}. \tag{2.6}$$

From Theorem Blundon we have

$$s^2 \le 2R^2+10Rr-r^2+2\sqrt{R\,(R-2r)^3}.$$

We denote $x=\dfrac{R}{r}$. It results that to find the best real number for which the inequality (2.6) is true (since it is known that the right side of inequality

Blundon is the best of the type $s^2 \leq f(R,r)$ where $f(R,r)$ is an homogeneous function). It will be sufficient to find the greatest real number t for which the following inequality is true.

$$2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3} \leq r^2 + 4Rr + \frac{18\left(8R^4 + tr^4\right)}{36R^2 - (16-t)r^2}, \quad (2.7)$$

inequality (2.7) may be written as

$$x^2 + 3x - 1 + \sqrt{x(x-2)^3} \leq \frac{9\left(8x^4 + t\right)}{36x^2 + t - 16}. \quad (2.8)$$

After performing some calculation we obtain

$$t \leq \frac{72x^4 - \left(36x^2 - 16\right)\left(x^2 + 3x - 1\right) - \left(36x^2 - 16\right)\sqrt{x(x-2)^3}}{x^2 + 3x - 10 + \sqrt{x(x-2)^3}}. \quad (2.9)$$

If we take $x = 2$ the inequality (2.8) becomes equality in the case of equilateral triangle. We consider $x \neq 2$.
The inequality (2.8) after dividing by $x - 2$ becomes

$$t \leq \frac{36x^3 - 36x^2 - 20x + 8 - \left(36x^2 - 16\right)\sqrt{x(x-2)}}{x + 5 + \sqrt{x(x-2)}} \quad (2.10)$$

for each $x > 2$.

We observe that the right side of inequality (2.10) is after use the conjugate and rationalize just the function F from statement.
From the inequality (2.10) it follows that the best real number is the minimum of the function on $(2, +\infty)$ interval.

After we will calculate F' we obtain:

$$F'(x) = \frac{1}{\left(x + 5 + \sqrt{x^2 - 2x}\right)^2 \sqrt{x^2 - 2x}}$$

$$\cdot \left[\sqrt{x^2 - 2x}\left(648x^2 - 360x - 108\right) - 648x^3 + 1008x^2 + 108x - 72\right] \quad (2.11)$$

The equation $F'(x) = 0$ may be written equivalent as:

$$\left(x^2 - 2x\right)\left(18x^2 - 10x - 3\right)^2 = \left(18x^3 - 28x^2 - 3x + 2\right)^2$$

or

$$18x^4 - 82x^3 - 4x^2 - 3x - 2 = 0. \quad (2.12)$$

From Descartes theorem it follows that the numbers of positive roots of equation (2.12) is less than or equal to 1.
We consider the polynomial function

$$g(x) = 18x^4 - 82x^3 - 4x^2 - 3x - 2.$$

Since $g(4)\,g(5) < 0$ it follows that equation $g(x) = 0$ has a unique root in (4,5) interval. But since $F'(4) < 0$ and $F'(5) > 0$ it follows that x_0 is a minimum point of function F on $(2, +\infty)$ interval. It result that the minimum of function F on $(2, +\infty)$ interval is $t_0 = F(x_0)$. In conclusion

$$a^2 + b^2 + c^2 \leq \frac{36\left(8R^4 + tr^4\right)}{36R^2 + (t-16)r^2} \text{ for each } t \in [-2, t_0].$$

We consider the function $h : [-2, t_0] \to R$

$$h(t) = \frac{36\left(t + 8x^4\right)}{t + 36x^2 - 16}$$

we have

$$h'(t) = \frac{-288\left(x^2 - 4\right)\left(x^2 - \frac{1}{2}\right)}{\left(t + 36x^2 - 16\right)^2} \text{ for each } t \in [-2, t_0].$$

It results that h is a decreasing function on $[-2, t_0]$. So, we have $h(t_0) \leq h(6) \leq h(t) \leq h(-2)$.

Finally we have the best chain of inequalities

$$a^2 + b^2 + c^2 \leq \frac{36\left(8R^4 + t_0 r^4\right)}{36R^2 + (t_0 - 16)r^2} \leq \frac{36\left(4R^4 + 3r^4\right)}{18R^2 - 5r^2}$$
$$\leq \frac{36\left(8R^4 + tr^4\right)}{36R^2 + (t - 16)r^2} \leq 8R^2 + 4r^2$$

for each $t \in [-2, 6]$.

In the following we give an irrational refinement of Gerretsen theorem. For this we find the best real numbers α, β, γ such that the inequality

$$a^2 + b^2 + c^2 \leq \sqrt{\alpha R^4 + \beta R^2 r^2 + \gamma r^4} \leq 8R^2 + 4r^2 \tag{2.13}$$

is true in any ABC triangle and the inequality

$$a^2 + b^2 + c^2 \leq \sqrt{\alpha R^4 + \beta R^2 r^2 + \gamma r^4}$$

is the best of this type.

In the case of the isosceles triangle with the sides $a = b = 1$, $c = 0$, $R = \frac{1}{2}$, $r = 0$, from (2.13) we obtain

$$\alpha = 64. \tag{2.14}$$

In the case of equilateral triangle from (2.13) we have

$$\frac{1}{12}\sqrt{1024 + 4\beta + \gamma} = 3$$

from where we obtain

$$4\beta + \gamma = 272. \tag{2.15}$$

From (2.13), (2.14) and (2.15) it follows that:

$$a^2 + b^2 + c^2 \leq \sqrt{64R^4 + \beta R^2 r^2 + (272 - 4\beta) r^4} \leq 8R^2 + 4r^2. \tag{2.16}$$

We denote $x = \frac{R}{r}$. From the inequality (2.16) it follows that

$$64x^4 + \beta x^2 + 272 - 4\beta \leq 64x^4 + 64x^2 + 16$$

or in an equivalent form

$$\beta + \frac{272 - 4\beta}{x^2} \leq 64 \text{ for each } x \geq 2.$$

If we take $x \to \infty$ we obtain $\beta \leq 64$.

In the following we will find the best real number $\beta \leq 64$ for which the inequality $a^2 + b^2 + c^2 \leq \sqrt{64R^4 + \beta R^2 r^2 + (272 - 4\beta) r^4}$ is the best of this type. $\square$

Theorem 2.2. *In any ABC triangle the following inequality is true*

$$a^2+b^2+c^2 \le \sqrt{64R^4+\beta R^2r^2+(272-4\beta)\,r^4} \le 8R^2+4r^2 \qquad (2.17)$$

for each $\beta\in[\beta_0,64]$, *where* $\beta_0=f(x_0)$ *and* x_0 *is an unique real positive root of the equation*

$$4x^5-5x^4-82x^3-164x^2-14x-4=0,\ x_0\approx 5.91016$$

where function f *is defined by* $f:(2,+\infty)\to R$,

$$f(x)=\frac{2\sqrt{x(x-2)}\left(x^2+3x-1\right)-2x^3-4x^2+11x+8}{x+2}.$$

and $\beta_0\approx 2.16151$.

Proof. From the identity

$$a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right)$$

the left side of the inequality (2.17) becomes

$$2\left(s^2-r^2-4Rr\right)\le\sqrt{64R^4+\beta R^2r^2+(272-4\beta)\,r^4}. \qquad (2.18)$$

From the Theorem Blundon we have

$$s^2\le 2R^2+10Rr-r^2+2\sqrt{R\,(R-2r)^3}.$$

It follows that to find the best real number $\beta\le 64$ for which the inequality (2.18) is true it will be sufficient to find the best real number $\beta\le 64$ for which the following inequality is true:

$$2\left(2R^2+6Rr-2r^2+2\sqrt{R\,(R-2r)^3}\right)\le\sqrt{64R^4+\beta R^2r^2+(272-4\beta)\,r^4}. \qquad (2.19)$$

The inequality (2.19) may be written in an equivalent form as

$$x^2+3x-1+\sqrt{x\,(x-2)^3}\le\sqrt{4x^4+\frac{\beta}{16}x^2+\frac{68-\beta}{4}}. \qquad (2.20)$$

After perform some calculation the inequality (2.20) may be written as

$$\beta\ge\frac{2\sqrt{x(x-2)}\left(x^2+3x-1\right)+x\,(x-2)^2-3x^3+7x+8}{x+2} \qquad (2.21)$$

for each $x>2$.

If we consider the function from the statement from (2.21) it follows that $\beta\ge f(x)$ for each $x\ge 2$. So the best real number which we find is the maximum of the function f on the $(2,+\infty)$ interval, $\beta_0=\max\limits_{x\in(2,+\infty)} f(x)$.

We have:

$$f'(x)=\frac{1}{(x+2)\sqrt{x^2-2x}}\cdot\left[4x^4+12x^3-2x^2-42x+4-\left(4x^3+16x^2+16x-14\right)\sqrt{x^2-2x}\right].$$

The equation $f'(x)=0$ is equivalent with:

$$\left(2x^4+6x^3-x^2-21x+2\right)^2=\left(2x^3+8x^2+8x-7\right)^2\left(x^2-2x\right)$$

or

$$4x^5-5x^4-82x^3-164x^2-14x-4=0. \qquad (2.22)$$

Since we have just one change of sign of the coefficients of the equation (2.22), from the Descartes Theorem it follows that the equation (2.22) has at most a positive root.

We consider the polynomial function

$$g\left(x\right)=4x^{5}-5x^{4}-82x^{3}-164x^{2}-14x-4.$$

Since $g\left(5\right)\cdot g\left(6\right)<0$ it follows that $x_0\in(5,6)$ is the only positive root of the equation (2.22). But since $f'\left(5\right)>0$ and $f'\left(6\right)<0$ it follows that x_0 is a maximum point of f. $\square$

Remark 2.1. We observe that $\beta=35$ is the best natural number for which the inequality (2.17) is true. So in any ABC triangle the following inequality is true:

$$a^2+b^2+c^2\le\sqrt{64R^4+35R^2r^2+132r^4}\le 8R^2+4r^2.$$

Theorem 2.3. *In any ABC triangle the following chain of inequalities is true:*

$$\begin{aligned}a^2+b^2+c^2&\le\sqrt{64R^4+\beta_0R^2r^2+\left(272-4\beta_0\right)r^4}\\ &\le\sqrt{64R^4+35R^2r^2+132r^4}\le\frac{8\left(27R^4+2r^4+8Rr^3\right)}{27R^2-8r^2}\\ &\le\frac{\left(4R+r\right)\left(4R^2-3Rr+2r^2\right)}{2R-r}\le\frac{36\left(4R^4+r^4\right)}{18R^2-7r^2}\le\frac{36\left(8R^4+r^4\right)}{36R^2-15r^2}\\ &\le\frac{72R^4}{9R^2-4r^2}\le 8R^2+4r^2\end{aligned}$$

where $\beta_0\in(34.49;34.58)$ *is the positive real number from Theorem* 2.2.

Proof. The inequality $\dfrac{(4R+r)(4R^2-3Rr+2r^2)}{2R-r}\le\dfrac{36(4R^4+r^4)}{18R^2-7r^2}$ is equivalent with $(x-2)(22x^2-48x+11)\ge0$ for each $x\ge2$ inequality which is true since the roots of equation $22x^2-48x+11=0$ are lower than 2.

It remain to prove that

$$\begin{aligned}\sqrt{64R^4+\beta_0R^2r^2+(272-4\beta_0)r^4}&\le\sqrt{64R^4+35R^2r^2+132r^2}\\ &\le\frac{8(27R^4+2r^4+8Rr^3}{27R^2-8r^2}\end{aligned}\tag{2.23}$$

the rest of inequalities follows from Theorem 2.2 and from introductive part.

To prove the left side of inequality (2.23), let $u=\dfrac{R^2}{r^2}$ and consider the function $h:[\beta_0,64]\to\mathbb{R}$, $h(\beta)=r^2\sqrt{64u^2+\beta u+272-4\beta}$. Note that

$$h'(\beta)=\frac{r^2(u-4)}{2\sqrt{64u^2+\beta u+272-4\beta}}\ge0.$$

It results that h is an increasing function. We have $h(\beta_0)\le h(\beta)$ for each $\beta\in[\beta_0,64]$. So, $h(\beta_0)\le h(35)$.

The left side of inequality (2.23) may be written as

$$\sqrt{64x^4+35x^2+132}\le\frac{8(27x^4+8x+2)}{27x^2-8}$$

for each $x \geq 2$ or after performing some calculation

$$2133x^6 + 27648 - 78292x^4 + 58880x^2 + 2048x - 81192 \geq 0$$

for each $x \geq 2$ or

$$(x-2)(2133x^5 + 31914x^4 - 14464x^3 - 28928x^2 + 1024x + 4096) \geq 0$$

for each $x \geq 2$, inequality which is true since
$2133x^5 + 31914x^4 - 14464x^3 - 28928x^2 + 1024x + 4096 = 2133x^2(x^3 - 8) + 7232x^3(x-2) + 2966x^2(x^2-4) + 21716x^4 + 1024x + 4096 \geq 0$ for each $x \geq 2$.

In the following we use the notation:

$U = \sqrt{64R^4 + 35R^2r^2 + 132r^4}$, $V = \dfrac{36(4R^4 + 3r^4)}{18R^2 - 5r^2}$,

$S = \sqrt{64R^4 + \beta_0 R^2r^2 + (272 - 4\beta_0)r^4}$, $T = \dfrac{36(8R^4 + t_0r^4)}{36R^2 + (t_0 - 16)r^2}$,

$W = \dfrac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 - 8r^2}$, $\gamma_0 = \sqrt{\dfrac{1561 + 8\sqrt{39544}}{90}}$

$A_0 = 73728 - 4608t_0 - 1296\beta_0$, $B_0 = -64t_0^2 + 4352t_0 - 73984 + 1152\beta_0 - 72\beta_0 t_0$,

$C_0 = -256t_0^2 - 2176t_0 + 17408 - \beta_0 t_0^2 + 32\beta_0 t_0 - 256\beta_0$, $\theta_0 = \sqrt{\dfrac{-B_0 + \sqrt{B_0^2 - 4A_0C_0}}{2A_0}}$,

where t_0 and β_0 was defined in Theorem 2.1 and Theorem 2.2. □

Theorem 2.4. *In any ABC triangle the following chains of inequalities is true:*

i) $a^2 + b^2 + c^2 \leq S \leq U \leq V \leq W$ *if* $\dfrac{R}{r} \geq \gamma_0$ *or* $a^2 + b^2 + c^2 \leq T \leq V < U \leq W$ *if* $2 \leq \dfrac{R}{r} < \gamma_0$*;*

ii) $a^2 + b^2 + c^2 \leq S \leq T \leq V \leq W$ *if* $\dfrac{R}{r} \geq \theta_0$ *or* $a^2 + b^2 + c^2 \leq T < S \leq U \leq W$ *if* $2 \leq \dfrac{R}{r} < \theta_0$.

Proof. i) First we shall prove that $V \leq W$ for every ABC triangle or

$$\frac{36(4x^4 + 3)}{18x^2 - 5} \leq \frac{8(27x^4 + 8x + 2)}{27x^2 - 8}$$

or after perform some calculation $4(x-2)(18x^3 + 324x^2 - 9x - 98) \geq 0$ which is true for each $x \geq 2$.

In the following we search the values of $\dfrac{R}{r}$ for which $U \leq V$ or

$$\sqrt{64R^4 + 35R^2r^2 + 132r^4} \leq \frac{36(4R^4 + 3r^4)}{18R^2 - 5r^2}. \tag{2.24}$$

If we denote $\dfrac{R^2}{r^2} = y$ the inequality (2.24) may be written as

$$\sqrt{64y^2 + 35y + 132} \leq \frac{36(4y^2 + 3)}{18y - 5}, \quad \text{for each } y \geq 4,$$

or after perform some calculation $(y-4)(180y^2-6244y-2091)\geq 0$ for each $y\geq 4$ or $y\geq \gamma_0^2$. It results that $U\leq V$ if $\frac{R}{r}\geq \gamma_0$ and $U>V$ if $2\leq \frac{R}{r}<\gamma_0$. The inequality $S\leq U$ was proved in Theorem 2.3. Also the inequality $T\leq V$ it follows from the monotony of function $f:\mathbb{R}\to\mathbb{R}$, $f(t)=\dfrac{36(8R^4+tr^4)}{36R^2+(t-16)r}$ since $f(t_0)\leq f(6)$.

ii) We search the values of $\frac{R}{r}$ for which $S\leq T$ or

$$\sqrt{64y^2+\beta_0 y+272-4\beta_0}\leq \frac{36(8y^2+t_0)}{36y+t_0-16}$$

or after square and perform some calculation $(y-4)(A_0y^2+B_0y+C_0)\geq 0$ for each $y\geq 4$.
Obviously $A_0>0$, $B_0<0$, $C_0<0$. So, we have $y\geq\theta_0^2$. It results that $S\leq T$ if $\frac{R}{r}>\theta_0$ and $S>T$ if $2\leq\frac{R}{r}<\theta_0$. □

Remark 2.2. The chain of inequalities from Theorem 2.4 may be complete after the term W with the rest of inequalities from Theorem 2.3.

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