The Blundon theorem in an acute triangle and some consequences

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ABSTRACT. The purpose of this article is to give an analoque of Blundon theorem in an acute triangle and using this result to obtain the best inequality of the type

$$\sum\sqrt{\frac{b+c-a}{a}}\geq f(R,r)$$

where f is a homogenous function

Let be $C(O,r), C(I,r)$ two circles such that $I\in int\ C(O,r)$ and $OI=\sqrt{R^2-2Rr}$.
For any triangle ABC with $C(O,R)$ the circumscrible and $C(I,r)$ the incircle, we denote $a=BC$, $b=CA$, $c=AB$, $s=\frac{a+b+c}{2}$ the semiperimeter of triangle and F the area.
The Theorem 2 of Blundon see [[3], p. 615-626] it has in this paper an analoque in an acute triangle by Theorem 3.
Also the Theorem 4 represent the best improvement of the type $\sum\sqrt{\frac{b+c-a}{a}}\geq f(R,r)$, where $f(R,r)$ is a homogeneous function of the inequality $\sum\sqrt{\frac{b+c-a}{a}}\geq 3$. See [[1], p. 159-165], which is know as the Rădulescu - Maftei Theorem and which in [1] has 2 solutions one elementary and other based on the multiplier Lagrange Theorem.

MAIN RESULTS

Lemma 1. In any triangle ABC are true the following equalities
1). $a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right)$
2). $ab+bc+ca=s^2+r^2+4Rr$
3). $a^2b^2+b^2c^2+c^2a^2=\left(s^2+r^2+4Rr\right)^2-16Rrs^2$

Lemma 2. In any triangle ABC is true the following equality:

$$\prod\cos A=\frac{s^2-r^2-4Rr-4R^2}{4R^2}$$

Proof. In the following we will denote $x=a^2+b^2+c^2$. From the cosine theorem it follows that:

$$\prod\cos A=\frac{\prod\left(b^2+c^2-a^2\right)}{8(\prod a)^2}=\frac{\prod\left(x-2a^2\right)}{8(\prod a)^2}=$$

$$=\frac{x^2-2\sum a^2x+4\sum a^2b^2x-8(\prod a)^2}{8(\prod a)^2}=\frac{s^2-r^2-4Rr-4R^2}{4R^2}$$

Theorem 1. In any acute triangle is true the following inequality:

$$s>2R+r$$

Proof. As in any acute triangle is true the inequality: $\prod\cos A>0$ according with Lemma 2 it follows the inequality from the statement.

Theorem 2. (Blundon). In any triangle ABC is true the following inequality: $s_1\leq s\leq s_2$ where

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$$s_1 = \sqrt{2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}}\ ,\ s_2 = \sqrt{2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3}}$$

represent the semiperimeter of two issoscels triangle $A_1B_1C_1$ and $A_2B_2C_2$ with the sides

$$a_1 = 2\sqrt{R^2 - (r-t)^2},\ b_1 = c_1 = \sqrt{2R(R+r-t)}$$

$$a_2 = 2\sqrt{R^2 - (r+t)^2},\ b_2 = c_2 = \sqrt{2R(R+r+t)}$$

where $t = OI = \sqrt{R^2 - 2Rr}$.

Lemma 3. Let $A_3B_3C_3$ be a triangle with $C(O,R)$ the circumscrible and $C(I,r)$ the incircle and with the semiperimeter $s_3 = 2R + r$. Then the sides of triangle $A_3B_3C_3$ is unique determinated by the equalities:

$$a_3 = 2R$$

$$b_3 = R + r + \sqrt{R^2 - 2Rr - r^2}$$

$$c_3 = R + r - \sqrt{R^2 - 2Rr - r^2}$$

where A_3 is a right angle.

Proof. We have the following equalities:

$$a + b + c = 2s$$

$$ab + bc + ca = s^2 + r^2 + 4Rr$$

$$abc = 4Rrs$$

or

$$a + b + c = 4R + 2r$$

$$ab + bc + ca = 4R^2 + 8Rr + 2r^2 \tag{1}$$

$$abc = 4Rr(2R + r)$$

From (1) it follows that a, b, c are the solutions of the equation:

$$u^3 - (4R + 2r)u^2 + \left(4R + 8Rr + 2r^2\right)u - 4Rr(2R + r) = 0 \tag{2}$$

The equation (2) may be written as:

$$(u - 2R)\left[u^2 - (2R + 2r)u + 4Rr + 2r^2\right] = 0$$

which has the solutions from the statement.

Theorem 3. In any acute triangle with $C(O,R)$ the circumscribed and $C(I,r)$ the inscribed are true the following inequalities:

$$s_1 \le s \le s_2 \text{ if } 2 \le \frac{R}{r} < \sqrt{2} + 1$$

and

$$s_3 \le s \le s_2 \text{ if } \frac{R}{r} \ge \sqrt{2}+1$$

where s_1, s_2 are the semiperimeter of two issosceles triangle $A_1B_1C_1, A_2B_2C_2$ with the sides from Theorem 2 and s_3 is the semiperimeter of the right triangle $A_3B_3C_3$ from Lemma 3.
Proof. We denote $\frac{R}{r} = x$. We consider two cases:
Case 1. $2 \le x < \sqrt{2}+1$
We will prove that $s_1 > s_3$ or in an equivalent form:

$$2x^2 + 10x - 1 - 2\sqrt{x(x-2)^3} - (2x+1)^2 = 2\left[-\sqrt{x(x-2)^3} - \left(x^2-3x+1\right)\right] > 0$$

or

$$-\left(x^2-3x+1\right) > \sqrt{x(x-2)^3} \tag{3}$$

But $x^2 - 3x + 1 < 0$ as $x < \sqrt{2}+1 < \frac{3+\sqrt{5}}{2}$. After squaring in (3) we obtain:

$$\left(x^2-3x+1\right)^2 > x(x-2)^3 \text{ or } -x^2+2x+1 > 0 \text{ or}$$

$$\left(\sqrt{2}-1-x\right)\left(x-\left(\sqrt{2}+1\right)\right) > 0$$

inequality which is true. It results that $s_3 < s_1 \le s_2$.
But as $s_1 \le s \le s_2$ and $s \ge s_3$ it follows that $s_1 \le s \le s_2$.
Case 2a. $\sqrt{2}+1 \le x < \frac{3+\sqrt{5}}{2}$ or $x^2-3x+1 < 0$.
We will prove that $s_1 \le s_3$ or in an equivalent form:

$$2x^2 + 10x - 1 - 2\sqrt{x(x-2)^3} < (2x+1)^2 \text{ or } -\left(x^2-3x+1\right) \le \sqrt{x(x-2)^3} \tag{4}$$

After squaring and performing some calculation the inequality (4) may be written as

$$\left(x-\left(\sqrt{2}-1\right)\right)\left(x-\left(\sqrt{2}+1\right)\right) \ge 0$$

inequality which is true.
We will prove that $s_3 < s_2$ or in an equivalent form:

$$(2x+1)^2 < 2x^2 + 10x - 1 - 2\sqrt{x(x-2)^3} \quad \text{or} \quad x^2-3x+1 < \sqrt{x(x-1)^3} \tag{5}$$

The inequality (5) is true as $x^2-3x+1 < 0$. It results that $s_1 \le s_3 < s_2$. But as $s_1 \le s \le s_2$ and $s \ge s_3$ it follows that $s_3 \le s \le s_2$.
Case 2.b. $x \ge \frac{3+\sqrt{5}}{2}$ or $x^2-3x+1 \ge 0$.
We will prove that

$$s_1 < s_3 \text{ or } -\left(x^2-3x+1\right) < \sqrt{x(x-2)^3}$$

inequality which is true.
We will prove that

$$s_3 < s_2 \text{ or } x^2-3x+1 < \sqrt{x(x-2)^3}$$

or in an equivalent form

$$\left[x-\left(\sqrt{2}-1\right)\right]\left[x-\left(\sqrt{2}+1\right)\right] > 0$$

It results that $s_1 < s_3 < s_2$. But as $s_1 \le s \le s_2$ and $s \ge s_3$ it follows that $s_3 \le s \le s_2$.
It results in the cases 2a and 2b that $s_3 \le s \le s_2$ which is equivalent with the inequality from the statement.

Lemma 4. In any triangle ABC is true the equalities:
1). $\sum \frac{s-a}{a} = \frac{s^2+r^2-8Rr}{4Rr}$
2). $\sum \frac{(s-a)(s-b)}{ab} = \frac{2R-r}{2R}$
Proof.

$$\sum \frac{s-a}{a} = \frac{s\sum bc - 3abc}{abc} = \frac{s\left(s^2+r^2+4Rr\right)-12Rr}{abc} = \frac{s^2+r^2-8Rr}{4Rr} =$$

$$= \sum \frac{(s-a)(s-b)}{ab} = \frac{s^2\left(\sum a\right) - 2s\left(s^2+r^2+4Rr\right)+12Rrs}{abc} = \frac{2R-r}{2R}$$

Theorem 4. (A refinement of Rădulescu - Maftei Theorem). In any triangle ABC is true the following inequality:

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{2R-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{r}{R}}$$

if $\frac{R}{r} \geq \sqrt{2}+1$ or

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{R-r-d}{r}} + 2\sqrt{\frac{R+d}{R}}$$

if $2 \leq \frac{R}{r} < \sqrt{2}+1$.

Proof. We denote $t = \sum \sqrt{\frac{s-a}{a}}$. By squaring we obtain

$$t^2 = \sum \frac{s-a}{a} + 2\sqrt{\frac{\sum (s-a)(s-b)}{ab} + 2\sqrt{\frac{(s-a)(s-b)(s-c)}{abc}}}$$

From Lemma 4, 1) and 2) it follows that:

$$\left(t^2 - \frac{s^2+r^2-8Rr}{4Rr}\right)^2 = 4\left(\frac{2R-r}{2R} + 2\sqrt{\frac{r}{4R}}t\right)$$

We consider the function $f : (0, +\infty) \to R$

$$f(u) = u^4 - \frac{s^2+r^2-8Rr}{2Rr}u^2 - 8\sqrt{\frac{r}{4R}}u + \left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2 - \frac{4R-2r}{R}$$

We have $f(t) = 0$. We will prove that

$$\left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2 < \frac{4R-2r}{R}$$

or in an equivalent form:

$$s^2 < 8Rr - r^2 + 4\sqrt{Rr^2(4R-2r)}$$

But as $s^2 \leq s_2^2$. It will be sufficient to prove that

$$s_2^2 = 2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3} < 8Rr - r^2 + 4\sqrt{Rr^2(4R-2r)} \tag{6}$$

We denote $x = \frac{R}{r}$. The inequality (6) may be written as:

$$2x^2 + 10x - 1 - 2\sqrt{x(x-2)^3} < 8x - 1 + 4\sqrt{x(4x-2)}$$

or

$$x^2 + x < \sqrt{x(x-2)^3} + 2\sqrt{x(4x-2)} \tag{7}$$

After squaring the inequality (7) we will obtain:

$$x^4+2x^3+x^2<x\left(x^3-6x^2+12x-8\right)+16x^2-8x+4x\sqrt{(x-2)^3(4x-2)}$$

or

$$8x^3-27x^2+16x<4x\sqrt{(x-2)^3(4x-2)}$$

or

$$8x^2-27x+16<4\sqrt{\left(x^3-6x^2+12x-8\right)(4x-2)} \tag{8}$$

If

$$8x^2-27x+16\leq 0$$

the inequality (8) is true. For $8x^2-27x+16>0$ we will square (8) and we will obtain:

$$64x^4+729x^2+256-432x^3+256x^2-864x<64x^4-416x^3+960x^2-896x+256$$

or

$$16x^3-25x^2-32x>0 \text{ or } 16x^2-25x-32>0$$

But $8x^2-27x+16>0$. It results that $x>\frac{27+\sqrt{217}}{16}>\frac{25+\sqrt{2673}}{32}$ or $16x^2-2x-32>0$.
We denote $a_2=\frac{s^2+r^2-8Rr}{2Rr}$, $a_1=8\sqrt{\frac{r}{4R}}$, $a_0=\frac{4R-2r}{R}-\left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2$. The equation $f(u)=0$ may be written as: $u^4-a_2u^2-a_1u-a_0=0$ with $a_0,a_1,a_2>0$ or $1-\frac{a_2}{u^2}-\frac{a_1}{u^3}-\frac{a_0}{u^4}=0$. But $g:(0,+\infty)\to R$, $g(u)=1-\frac{a_2}{u^2}-\frac{a_1}{u^3}-\frac{a_0}{u^4}$ is an increasing function. It results that t is the only positive root of equation $f(u)=0$.
It result that if exists a unique continue function $u:[s_1,s_2]\to R$ such that $f(u(s))=0$, $(\forall)\, s\in[s_1,s_2]$. From implicite Theorem it follows that u is derivable on interval (s_1,s_2), $u:[s_1,s_2]\to R$ which verify the condition:

$$\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)^2=4\left(\frac{2R-r}{2R}+2\sqrt{\frac{r}{4R}}u(s)\right),\quad (\forall)\, s\in[s_1,s_2] \tag{9}$$

After we derivate the equality (9) we will obtain:

$$\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)\left(u(s)u'(s)-\frac{s}{4Rr}\right)=\sqrt{\frac{r}{R}}u'(s),(\forall)\, s\in[s_1,s_2]$$

or in an equivalent form:

$$\left(u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}\right)u'(s)=\frac{s}{4Rr}\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)$$

or

$$\left(u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}\right)u'(s)=\frac{s}{4Rr}\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right),(\forall)\, s\in[s_1,s_2]$$

From:

$$u^2(s)=\sum\frac{s-a}{a}+2\sum\sqrt{\frac{(s-a)(s-b)}{ab}}\geq\frac{s^2+r^2-8Rr}{4Rr}+6\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{abc}}=$$

$$=\frac{s^2+r^2-8Rr}{4Rr}+6\sqrt[3]{\frac{r}{4R}},(\forall)\, s\in[s_1,s_2]$$

it results that:

$$u^3(s) - \frac{s^2+r^2-8Rr}{4Rr}u(s) - \sqrt{\frac{r}{R}} = u(s)\left(u^2(s) - \frac{s^2+r^2-8Rr}{4Rr}\right) - \sqrt{\frac{r}{R}} \geq \sqrt{6\sqrt[3]{\frac{r}{4R}}\cdot 6\sqrt[3]{\frac{r}{4R}}} -$$

$$-\sqrt{\frac{r}{R}} = \left(3\sqrt{6}-1\right)\sqrt{\frac{r}{R}} > 0, \ \ (\forall)\, s \in [s_1, s_2]$$

and $u^2(s) - \frac{s^2+r^2-8Rr}{4Rr} > 0$, $(\forall)\, s \in [s_1, s_2]$. It results that u is an increasing function on interval $[s_1, s_2]$. From Theorem 3 it follows that $s_1 \leq s$, for $2 \leq \frac{R}{r} < \sqrt{2}+1$ which implies that $u(s_1) \leq u(s)$. Replacing the sides a_1, b_1, c_1 of the $A_1B_1C_1$ triangle from Theorem 2 we will obtain:

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{R-r-d}{r}} + 2\sqrt{\frac{R+d}{R}} \text{ if } 2 \leq \frac{R}{r} < \sqrt{2}+1$$

From Theorem 3 it follows that $s_3 \leq s$ if $\frac{R}{r} \geq \sqrt{2}+1$ which implies that $u(s_3) \leq u(s)$
By replacing the sides a_3, b_3, c_3 from Lemma 3 it follows that:

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{2R^2 - 2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{r}{R}} \text{ if } \frac{R}{r} \geq \sqrt{2}+1$$

Lemma 5. In any triangle ABC is true the following inequality:

$$\sqrt{\frac{2R - 2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{r}{R}} \geq 3 \text{ if } \frac{R}{r} \geq \sqrt{2}+1 \tag{10}$$

Proof. We denote $d_2 = \sqrt{x^2-2x-1}$. By squaring the inequality (10) we will obtain:

$$2\sqrt{\frac{(2x-2d_2)(2x+2d_2)}{(x+1+d_2)(x+1-d_2)}} + \frac{(2x-2d_2)(x+1-d_2)+(2x+2d_2)(x+1+d_2)}{(x+1+d_2)(x+1-d_2)} \geq \left(3 - \frac{1}{\sqrt{x}}\right)^2$$

or

$$2\sqrt{\frac{4(x^2-x^2+2x+1)}{x^2+2x+1-x^2+2x+1}} +$$

$$+\frac{2x^2+2x-2xd_2-2xd_2-2d_2+2x^2-4x-2+2x^2+2x+2xd_2+2xd_2+2d_2+2x^2-4x-2}{x^2+2x+1-x^2+2x+1} \geq$$

$$\geq 9 + \frac{1}{x} - \frac{6}{\sqrt{x}}$$

or

$$\frac{8x^2-4x-4}{4x+2} + 2\sqrt{2} \geq 9 + \frac{1}{x} - \frac{6}{\sqrt{x}}$$

or

$$2x - 2 + 2\sqrt{2} \geq 9 + \frac{1}{x} - \frac{6}{\sqrt{x}}$$

or

$$2x - 11 + 2\sqrt{2} \geq \frac{1}{x} - \frac{6}{\sqrt{x}}$$

or

$$2x^2+\left(2\sqrt{2}-11\right)x\geq 1-6\sqrt{x}$$

or

$$2x^2+\left(2\sqrt{2}-11\right)x+6\sqrt{x}-1\geq 0$$

We consider the function $f:\left[\sqrt{2}+1,+\infty\right)\rightarrow R$

$$f\left(x\right)=2x^2+\left(2\sqrt{2}-11\right)x+6\sqrt{x}-1$$

with the derivate

$$f'\left(x\right)=4x+2\sqrt{2}-11+\frac{3}{\sqrt{x}}=4\left(x-\sqrt{2}-1\right)+6\sqrt{2}-7+\frac{3}{\sqrt{x}}\geq 0$$

It results that f is an increasing function on interval $\left[\sqrt{2}+1,+\infty\right)$ which implies that $f\left(x\right)>f\left(\sqrt{2}+1\right)$.

After performing some calculation we obtain $f\left(\sqrt{2}+1\right)>0$.

Lemma 6. In any triangle ABC is true the following inequality:

$$\sqrt{\frac{R-r-d}{r}}+2\sqrt{\frac{R+d}{R}}\geq 3,\ \text{if}\ 2\leq\frac{R}{r}\leq 8 \tag{11}$$

Proof. We denote $\frac{R}{r}=x, d_x=\frac{\sqrt{R(R-2r)}}{r}=\sqrt{x\left(x-2\right)}$. The inequality (11) may be written as:

$$\sqrt{x-1-d_x}+2\sqrt{\frac{x+d_x}{x}}\geq 3$$

By squaring we will obtain:

$$\frac{4x+4d_x}{x}\geq 9+x-1-d_x-6\sqrt{x-1-d_x}$$

or

$$6\sqrt{x-1-d_x}\geq 8-d_x+x-\frac{4x+4d_x}{x}$$

or

$$6\sqrt{x-1-d_x}\geq\frac{4x-xd_x+x^2-4d_x}{x}$$

or

$$6\sqrt{x-1-d_x}\geq\frac{\left(x+4\right)\left(x-d_x\right)}{x}$$

or

$$36x^2\left(x-1-d_x\right)\geq\left(x^2+8x+16\right)2\left(x-d_x-1\right)x$$

or

$$2x\left(x-d_x-1\right)\left(18x-x^2-8x-16\right)\geq 0 \text{ and as } x-d_x-1>0$$

It will be sufficient to prove that:

$$x^2-10x+16\leq 0 \text{ or } \left(x-2\right)\left(x-8\right)\leq 0 \text{ or } x\leq 8$$

Theorem 5. (The inequality Rădulescu-Maftei) In any acute triangle is true the following inequality:

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}\geq 3$$

Proof. It results from Theorem 4, Lemma 5 and 6.

Theorem 6. In any triangle ABC with $2\leq\frac{R}{r}\leq 8$ is true the following inequality:

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}\geq 3$$

Proof. According with the proof of Theorem 4 it follows that $u:[s_1,s_2]\to R$ is an increasing function. But $s_1\leq s$. It results that $u(s)\geq u(s_1)$ or

$$\sum\sqrt{\frac{b+c-a}{a}}\geq\sqrt{\frac{R-r-d}{r}}+2\sqrt{\frac{R+d}{R}}\geq 3$$

according with Lemma 6.

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