AN WEIGHTED POWER LESSELS-PELLING INEQUALITY

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ABSTRACT. Lessels-Pelling inequality states that in a triangle the sum of two angle bisectors and a medians is less or equal than the product between $\sqrt{3}$ and semiperimeter. The purpose of this article is to find a powered option of this inequality with some supplementary condition, for the weights and some chains of inequalities which represent refinements of powered Lessels-Pelling inequality.

1. INTRODUCTION

Let ABC be a triangle. We shall denote $a = BC$, $b = CA$, $c = AB$, $s = \frac{a+b+c}{2}$ the semiperimeter, m_a the medians from A, w_b, w_c the bisectors from B and C.

In 1974 J. Garfunkel [1] conjunctured on basis of computer check the following inequality $m_a + w_b + h_c \leq s\sqrt{3}$, where h_c represent the altitude from C.

In 1976 C.S. Gardner [1] prove by mean of sequence of elementary transformation and differentiable of a function.

In 1977 G.S. Lessels and M.J. Pelling give in [2] stronger inequality $m_a + w_b + w_c \leq s\sqrt{3}$ using the computer check.

In 1980 B.E. Patuwo, R.S.D. Thomas and Chung-Lie Wang give in [3] a proof of this inequality.

In [4] appear a new proof of C. Tănăsescu.

In 1981 L. Panaitopol [5] find an elementary solution of Lessels-Pelling inequality.

În [6] and [7] M. Drăgan give a simple proof and some refinements of Lessels-Pelling inequality.

2. MAIN RESULTS

In the following we use the following notation

$$x = \frac{a}{s},\ y = \frac{b}{s},\ z = \frac{c}{s},\ u = \sqrt{1-y},\ v = \sqrt{1-z},\ s_1 = u + v,\ p_1 = u \cdot v,$$
$$E = \sqrt{\frac{2(y^2+z^2)-x^2}{4}},\ s_2 = \beta u + \gamma v,\ t = \frac{3\alpha^2 - 2\alpha + 1}{2}, \tag{2.1}$$

where α, β, γ are positive numbers such that $(\beta-\gamma)(b-c) \geq 0$ and $\alpha+\beta+\gamma = 1$.

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Lemma 2.1. *According with above notation, are true the inequalities:*

$$i) \qquad E \le \sqrt{1 - \frac{s_1^2}{2}} \tag{2.2}$$

$$ii) \qquad E \le \sqrt{1 - \frac{2}{(1-\alpha)^2}\, s_2^2} \tag{2.3}$$

Proof. *i)* From (2.1) we have

$$\begin{aligned}
E &= \sqrt{\frac{2(2+u^4+v^4-2u^2-2v^2)-(u^2+v^2)^2}{4}} \\
&= \sqrt{\frac{2[2+(u^2+v^2)^2-2p_1^2-2(u^2+v^2)]-(u^2+v^2)^2}{4}} \\
&= \sqrt{\frac{4+2(s_1^2-2p_1)^2-4p_1^2-4(s_1^2-2p_1)-(s_1^2-2p_1)^2}{4}} \\
&= \sqrt{\frac{4+s_1^4-4s_1^2+4p_1(2-s_1^2)}{4}} \le \sqrt{\frac{4-4s_1^2+2s_1^2}{4}} = \sqrt{1-\frac{s_1^2}{2}}
\end{aligned}$$

ii) We have

$$s_2 = \beta u + \gamma v \le \frac{1}{2}\,(\beta+\gamma)(u+v) = \frac{(1-\alpha)s_1}{2} \tag{2.4}$$

since $(\beta-\gamma)(b-c) \ge 0$, it results that $(\beta-\gamma)(u-v) \le 0$.
From (2.2) and (2.4) it results (2.3). □

Theorem 2.1 (The weighted power Lessels-Pelling inequality)**.** *In every ABC triangle is true the following inequality*

$$\alpha m_a + \beta w_b + \gamma w_c \le S\sqrt{t} \tag{2.5}$$

where α, β, γ are positive numbers such that $(\beta-\gamma)(b-c) \ge 0$ and $\alpha+\beta+\gamma = 1$.

Proof. Since $w_b \le \sqrt{s(s-b)}$ and $w_c \le \sqrt{s(s-c)}$ to prove (2.5) it will be sufficient to prove that

$$\alpha m_a + \beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)} \le s\sqrt{t} \tag{2.6}$$

According (2.1) inequality, (2.6) may be written as

$$\alpha\sqrt{\frac{2(y^2+z^2)-x^2}{4}} + \beta\sqrt{1-y} + \gamma\sqrt{1-z} \le \sqrt{t}$$

or

$$\alpha E + S_2 \le \sqrt{t} \tag{2.7}$$

From (2.2) and (2.4) we have that

$$\alpha E + S_2 \le \alpha\sqrt{\frac{2-S_1^2}{2}} + \frac{1-\alpha}{2}\, S_1$$

It results that to prove (2.7), it will be sufficient to prove that

$$\alpha\sqrt{\frac{2-S_1^2}{2}} + \frac{1-\alpha}{2}\, S_1 \le \sqrt{t} \tag{2.8}$$

We note that $S_1 = \sqrt{1-y} + \sqrt{1-z} \leq \sqrt{2(2-y-z)} = \sqrt{2x} = \sqrt{\frac{2\alpha}{S}} < \sqrt{2}$.
So, we have $S_1 < \sqrt{2}$.
We consider the function $f : \left(0, \sqrt{2}\right) \to \mathbb{R}$, $f(S_1) = \alpha\sqrt{\frac{2-S_1^2}{2}} + \frac{(1-\alpha)S_1}{2}$ with $f'(S_1) = \frac{-\alpha S_1}{\sqrt{4-2S_1^2}} + \frac{1-\alpha}{2}$. We solve the equation $f'(S_1) = 0$ with the root $S_0 = \frac{2(1-\alpha)}{\sqrt{6\alpha^2 - 4\alpha + 2}}$.
Since $f'(0) \geq 0$, $f'\left(\sqrt{2}\right) \leq 0$, it results that S_0 is a maximum point for f. So we have

$$\begin{aligned} f(S_1) \leq f(z_0) &= \alpha\sqrt{\frac{1}{2}\left[2 - \frac{4(1-\alpha)^2}{6\alpha^2-4\alpha+2}\right]} + \frac{(1-\alpha)^2}{\sqrt{6\alpha^2-4\alpha+2}} \\ &= \alpha\sqrt{\frac{4\alpha^2}{6\alpha^2-4\alpha+2}} + \frac{(1-\alpha)^2}{\sqrt{6\alpha^2-4\alpha+2}} \\ &= \sqrt{\frac{3\alpha^2-2\alpha+1}{2}} = \sqrt{t} \end{aligned}$$

which prove (2.8).
So, we have the following refinement of (2.5). □

Theorem 2.2. *In every ABC triangle is true the following chain of inequalities:*

$$\begin{aligned} \alpha m_a + \beta w_b + \gamma w_c &\leq \alpha m_a + \beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)} \\ &\leq \alpha\sqrt{\frac{2s^2 - s\left(\sqrt{s-b}+\sqrt{s-c}\right)^2}{2}} \\ &+ \frac{1-\alpha}{2}\left(\sqrt{s(s-b)} + \sqrt{s(s-c)}\right) \leq s\sqrt{t} \end{aligned}$$

where α, β, γ are positive numbers such that $(\beta-\gamma)(b-c) \geq 0$ and $\alpha+\beta+\gamma = 1$.

Remark 2.1. If we take in (2.5) $\alpha = \beta = \gamma$ we obtain the Lessels-Pelling inequality.

Theorem 2.3. *In every ABC triangle are true the following inequalities:*

i)
$$m_a \leq \frac{\sqrt{t}}{2\alpha} \cdot \frac{m_a^2}{s} + \frac{\alpha}{2\sqrt{t}} s \tag{2.9}$$

ii)
$$\frac{\sqrt{t}}{2\alpha}\frac{m_a^2}{s} + \frac{\alpha}{2\sqrt{t}} s \leq \frac{t+\alpha^2}{2\alpha\sqrt{t}} s - \frac{\sqrt{t}}{\alpha(1-\alpha)^2}\left(\beta\sqrt{s-b} + \gamma\sqrt{s-c}\right)^2 \tag{2.10}$$

iii) $$m_a \le \sqrt{s^2 - \frac{2}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2}$$
$$\le \frac{t+\alpha^2}{2\alpha\sqrt{t}}\,s - \frac{\sqrt{t}}{\alpha(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2 \qquad (2.11)$$

where α, β, γ *are positive numbers such that* $\alpha+\beta+\gamma = 1$ *and* $(\beta-\gamma)(b-c) \ge 0$.

Proof. *i)* From (2.1), inequality (2.9) may be written as
$$E \le \frac{\sqrt{t}}{2\alpha}E^2 + \frac{\alpha}{2\sqrt{t}} \quad \text{or} \quad \left(\sqrt{t}E-\alpha\right)^2 \ge 0.$$

ii) From (2.3) we have
$$\frac{\sqrt{t}}{2\alpha}E^2 + \frac{\alpha}{2\sqrt{t}} \le \frac{\sqrt{t}}{2\alpha} - \frac{\sqrt{t}}{\alpha(1-\alpha)^2}S_2^2 + \frac{\alpha}{2\sqrt{t}}$$
$$= \frac{t+\alpha^2}{2\alpha\sqrt{t}} - \frac{\sqrt{t}}{\alpha(1-\alpha)^2}S_2^2 \qquad (2.12)$$

Taking in (2.12) the notation from (2.1) we obtain the equality from statement.

iii) The first inequality is just inequality (2.3). We will prove the second inequality which is equivalent with
$$\sqrt{1-\frac{2}{(1-\alpha)^2}S_2^2} \le \frac{t+\alpha^2}{2\alpha\sqrt{t}} - \frac{\sqrt{t}}{\alpha(1-\alpha)^2}S_2^2$$
or
$$\left[\sqrt{t-\frac{2t}{(1-\alpha)^2}S_2^2 y} - \alpha\right]^2 \ge 0. \qquad \square$$

Theorem 2.4. *In every ABC triangle is true the following chain of inequalities:*

i) $$\alpha m_a + \beta w_b + \gamma w_c \le \alpha m_a + \beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)}$$
$$\le \frac{\sqrt{t}}{2}\frac{ma^2}{s} + \frac{\alpha^2}{2\sqrt{t}}s + \beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)}$$
$$\le \frac{t+\alpha^2}{2\sqrt{t}}s - \frac{\sqrt{t}}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2$$
$$+\beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)} \le s\sqrt{t}$$

ii) $$\alpha m_a + \beta w_b + \gamma w_c \le \alpha\sqrt{s^2 - \frac{2}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2} + \beta w_b + \gamma w_c$$
$$\le \frac{t+\alpha^2}{2\sqrt{t}}s - \frac{\sqrt{t}}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2 + \beta w_b + \gamma w_c$$
$$\le \frac{t+\alpha^2}{2\sqrt{t}}s - \frac{\sqrt{t}}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2$$
$$+\beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)} \le s\sqrt{t},$$

where α, β, γ *are positive numbers such that* $\alpha + \beta + \gamma = 1$ *and* $(\beta - \gamma)(b - c) \geq 0$.

Proof. *i*) The first inequality it follows from inequalities $w_b \leq \sqrt{s(s-b)}$ and $w_c \leq \sqrt{s(s-c)}$.
The second inequality it results from (2.9). Also the third it follows from (2.10). It remain to prove that

$$\frac{t+\alpha^2}{2\sqrt{t}}\, s - \frac{\sqrt{t}}{2(1-\alpha)^2}\left(\beta\sqrt{s-b} + \gamma\sqrt{s-c}\right)^2$$
$$+\beta\sqrt{s(s-b)} + \gamma\sqrt{s(s-c)} \leq s\sqrt{t}. \tag{2.13}$$

Using (2.1), inequality (2.13) may be written as

$$\frac{t+\alpha^2}{2\sqrt{t}} - \frac{\sqrt{t}}{(1-\alpha)^2}\, S_2^2 + S_2 \leq \sqrt{t}$$

or

$$tS_2^2 - 2\sqrt{t}(1-\alpha)^2 S_2 + (t-\alpha^2)(1-\alpha)^2 \geq 0$$

or

$$2tS_2^2 - 2\sqrt{t}(1-\alpha)^2 S_2 + \frac{(1-\alpha)^4}{2} \geq 0$$

or

$$\left(2\sqrt{t}S_2 - 1 + \alpha\right)^2 \geq 0.$$

ii) The first inequality from this chain it follows from (2.11). The second it results also from (2.11). The third is true as $w_b \leq \sqrt{s(s-b)}$, $w_c \leq \sqrt{s(s-c)}$, and last inequality from chain is the inequality from *i*) □

Corollary 2.1. *In every ABC triangle is true the following inequality*

$$\alpha m_a + \beta w_b + \gamma w_c \leq \sqrt{\frac{2\alpha^2 + (\beta+\gamma)^2}{2}}\, s, \tag{2.14}$$

where α, β, γ *are positive numbers such that* $(b-c)(\beta-\gamma) \geq 0$.

Proof. If we take $\alpha \to \dfrac{\alpha}{\alpha+\beta+\gamma}$, $\beta \to \dfrac{\beta}{\alpha+\beta+\gamma}$, $\gamma \to \dfrac{\gamma}{\alpha+\beta+\gamma}$ in (2.5), we obtain (2.14). □

Corollary 2.2. *In every ABC triangle is true the following inequality*

$$m_a w_a + m_c w_b + m_b w_c \leq \sqrt{\frac{2w_a^2 + (m_b+m_c)^2}{2}}.$$

Proof. Since $(b-c)(m_c - m_b) \geq 0$, if we take $\alpha = w_a$, $\beta = m_c$, $\gamma = m_b$ in (2.14), we obtain the inequality from the statement. □

Corollary 2.3. *In every ABC triangle is true the following inequality:*

$$m_a m_b m_c + w_a w_b m_c + w_a m_b w_c \leq \sqrt{m_b^2 m_c^2 + \frac{w_a^2(m_b+m_c)^2}{2}}. \tag{2.15}$$

Proof. Since $(b-c)\left(\dfrac{1}{m_b} - \dfrac{1}{m_c}\right) \geq 0$, if we take in (2.14) $\alpha = \dfrac{1}{w_a}$, $\beta = \dfrac{1}{m_b}$, $\gamma = \dfrac{1}{m_c}$, we obtain the inequality from the statement. □

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