

# Geometric properties of the functions $\Gamma$ and $1/\Gamma$

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## Abstract

For an arbitrary positive real number  $r > 0$ , let  $U(r) = \{z \in \mathbb{C} : |z| < r\}$  denote the disk centered in 0 and of radius  $r$ . Let  $\mathcal{A}$  be the class of analytic functions defined in the unit disk  $U = U(1) = \{z \in \mathbb{C} : |z| < 1\}$  and having the property  $f(0) = f'(0) - 1 = 0$ . We say that a function  $f \in \mathcal{A}$  is starlike if  $f$  is univalent, and  $f(U)$  is a starlike domain with respect to 0, and a function  $f \in \mathcal{A}$  is convex if  $f$  is univalent and  $f(U)$  is a convex domain in  $\mathbb{C}$ . Analytic descriptions of these properties are as follows

$$f \in \mathcal{A} \text{ is starlike if and only if } \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U,$$

and

$$f \in \mathcal{A} \text{ is convex if and only if } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U.$$

If  $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ ,  $z \in U$  or  $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ ,  $z \in U$ , we say that  $f$  is starlike of order  $\alpha$  and convex of order  $\alpha$  respectively. We define by the equalities

$$r_f^*(\alpha) = \sup \left\{ r \in (0, \infty) : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U(0, r) \right\}$$

and

$$r_f^c(\alpha) = \sup \left\{ r \in (0, \infty) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U(0, r) \right\},$$

the radius of starlikeness and the radius of convexity of order  $\alpha$  respectively, where  $\alpha \in [0, 1)$ , is a given number and  $f \in \mathcal{A}$ , is a given function.

It is obvious that the mapping defined by ([1] p.3)

$$u(z) = \frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

belongs to  $\mathcal{A}$ . In [2] the authors determined the radius of starlikeness of  $u$ . We are going to generalize the result regarding the starlikeness and to determine the radius of convexity of the mapping  $u$ .

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## References

- [1] G.E. Andrews, R. Asky and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999
- [2] E.P. Merkes, M.S. Robertson, W.T. Scott; *On products of starlike functions*, Proceedings of the American Mathematical Society Vol. 13, No. 6 (Dec., 1962), pp. 960-964.

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