Modern Portfolio Theory

History of MPT

- 1952 Horowitz
- CAPM (Capital Asset Pricing Model)
  1965 Sharpe, Lintner, Mossin
- APT (Arbitrage Pricing Theory) 1976 Ross

What is a portfolio?

- Italian word
- Portfolio weights indicate the fraction of the portfolio total value held in each asset
- x_i = (value held in the i-th asset)/(total portfolio value)
- By definition portfolio weights must sum to one:

$$x_1 + x_2 + \ldots + x_{n-1} + x_n = 1$$

Data needed for Portfolio Calculation

- Expected returns for asset i : $E\left(r_i\right)$

- Variances of return for all assets i : $Var\left(r_i\right)$

- Covariances of returns for all pairs of assets I

  and j : $Cov\left(r_i, r_j\right)$

Where do we obtain this data?

- Compute them from knowledge of the probability distribution of returns (*population parameters*)
- Estimate them from historical sample data using statistical techniques (*sample statistics*)

Examples

Market Economy	Probability	Return
Normal environment	1:3	10%
Growth	1:3	30%
Recession	1:3	-10%

$$E(r) = 1/3(0,30) + 1/3(0,10) + 1/3(-0,10) = 0,10$$

$$Var(r) = 1/3(0,30 - 0,10)^2 + 1/3(0,10 - 0,10)^2 + 1/3(-0,10 - 0,10)^2 =$$

$$= 1/3(0,20)^2 + 1/3(0,0)^2 + 1/3(-0,20)^2 = 0,0267$$

Portfolio of two assets(1)

- The portfolio's expected return is a weighted sum of the expected returns of assets 1 and 2.

$$E\left(r_v\right) = E\left(w_1 r_1\right) + E\left(w_2 r_2\right) = w_1 E\left(r_1\right) + w_2 E\left(r_2\right)$$

$$Var(r_v) = E\left(r_v^2\right) + E(r_v)^2 = w_1^2\left[E\left(r_1^2\right) - E\left(r_1\right)^2\right] +$$

$$+ w_2^2\left[E\left(r_2^2\right) - E\left(r_2\right)^2\right] + 2w_1 w_2\left[E\left(r_1 r_2\right) - E\left(r_1\right)E\left(r_2\right)\right] =$$

$$= w_1^2 Var\left(r_1\right) + w_2^2 Var\left(r_2\right) + 2w_1 w_2 Cov\left(r_1, r_2\right)$$

Portfolio of two assets(2)

- The variance is the square-weighted sum of the variances plus twice the cross-weighted covariance.
- If

$$\mu_v = E(r_v),\ \sigma_v = \sqrt{Var(r_v)},\ \rho_{1,2} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2}$$

then

Where $\rho_{1,2}$ is the corellation

$$\mu_v = w_1 \mu_1 + w_2 \mu_2$$

$$\sigma_v^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{1,2} \sigma_1 \sigma_2$$

Portfolio of Multiple Assets(1)

- We can write weights in form of matrix $1 = uw^T$

- also the expected returns can be write in form of vector $m = [\mu_1, \mu_2, \ldots, \mu_n]$

- and let C the covariance matrix

$$C = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix}$$

where $c_{i,j} = Cov(r_i, r_j)$

Portfolio of Multiple Assets(2)

- Because C is symmetric then $\exists \quad C^{-1}$

- Then the expected return is equal with:

$$\mu_v = mw^T$$

- Variance of returns is equal with:

$$\sigma_v^2 = wCw^T$$

Proof

$$\mu_v = E(r_v) = E\left(\sum_i w_i r_i\right) = \sum_i w_i q \mu_i = m w^T$$

$$\sigma_v^2 = Var(r_v) = Var\left(\sum_i w_i r_i\right) = Cov\left(\sum_i w_i r_i \sum_j w_j r_j\right) =$$

$$= \sum_{i,j} w_i w_j c_{i,j} = w C w^T$$

Correlation

$$\sigma_v = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \ldots + w_n^2\sigma_n^2 + \sum_{i,j}^{n,n} 2\rho_{i,j} w_i\sigma_i w_j\sigma_j}$$

$$if \quad \underline{\underline{\rho_{i,j} = 1}} \; \forall\, i, j \in \overline{i,n}$$

$$\sigma_v = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \ldots + w_n^2\sigma_n^2 + \sum_{i,j}^{n,n} 2 w_i\sigma_i w_j\sigma_j}$$

$$\sigma_v = w_1\sigma_1 + w_2\sigma_2 + \ldots + w_n\sigma_n = \sigma_{v\,gen}$$

Correlation(2)

- An equally-weighted portfolio of *n* assets:

$$w_i = \frac{1}{n} \quad \sigma_i = \sigma_{gen}$$

$$\sigma_v = \left(\frac{1}{n}\sigma_{gen}\right)_1 + \ldots + \left(\frac{1}{n}\sigma_{gen}\right)_n$$

$$\sigma_v = n\frac{1}{n}\sigma_{gen} = \sigma_{gen}$$

If the correlation is equal with 1 then between i and j is linear connection; if i grow then j grow to and growth rate is the same

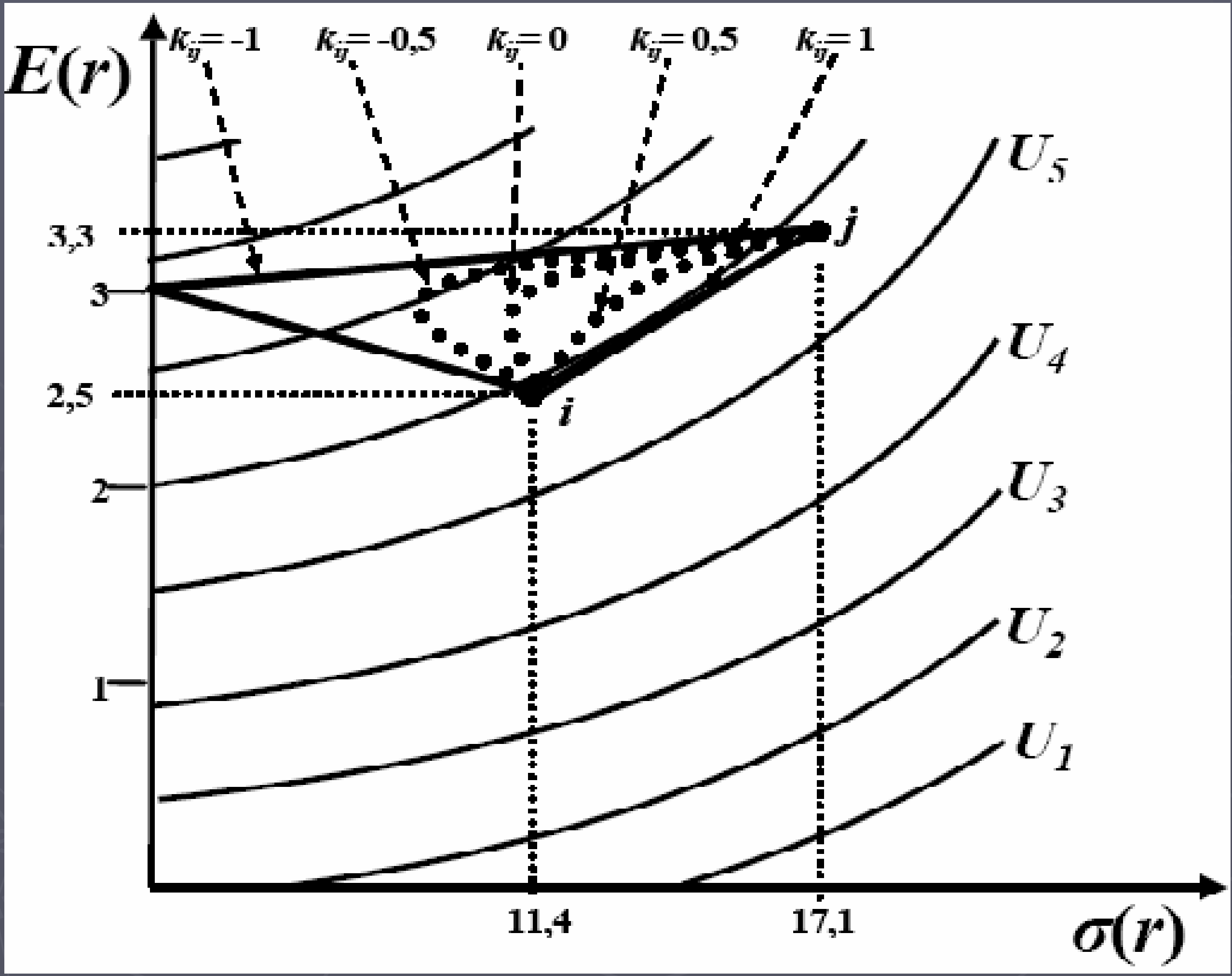
Correlation(3)

$$if\ \underline{\underline{\rho_{i,j}=0}} \quad w_i=\frac{1}{n} \quad \sigma_i=\sigma_{gen}$$

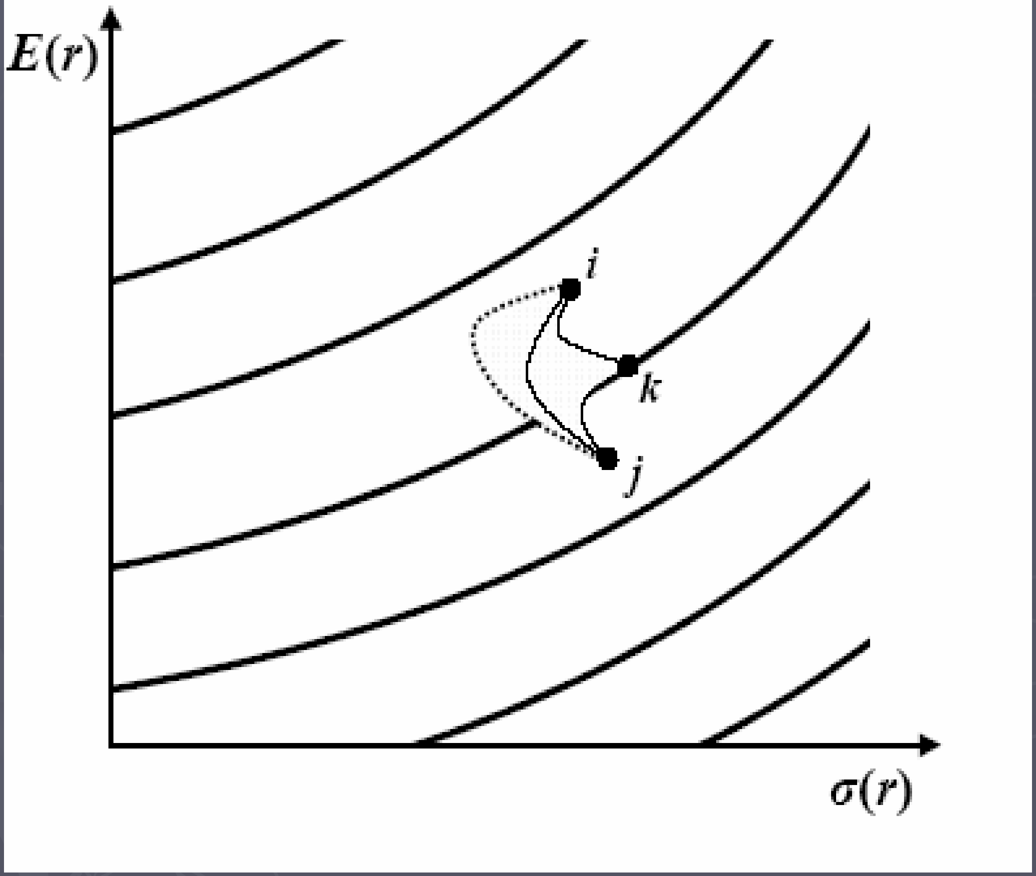
$$\sigma_v=\sqrt{\left(\frac{1}{n}\sigma_{gen}\right)_1^2+\ldots+\left(\frac{1}{n}\sigma_{gen}\right)_n^2}$$

$$\sigma_v=\sqrt{n\left(\frac{1}{n}\sigma_{gen}\right)^2}\Rightarrow\sigma_v=\frac{\sqrt{n}}{n}\sigma_{gen}$$

Diversification



Diversification(2)



If we have 3 element in our portfolio than the variance of portfolio is much lower

Diversification(3)

- Reducing risk with this technique is called diversification
- Generally the more different the assets are, the greater the diversification.
- The diversification effect is the reduction in portfolio standard deviation, compared with a simple linear combination of the standard deviations, that comes from holding two or more assets in the portfolio
- The size of the diversification effect depends on the degree of correlation

Optimal portfolio selection

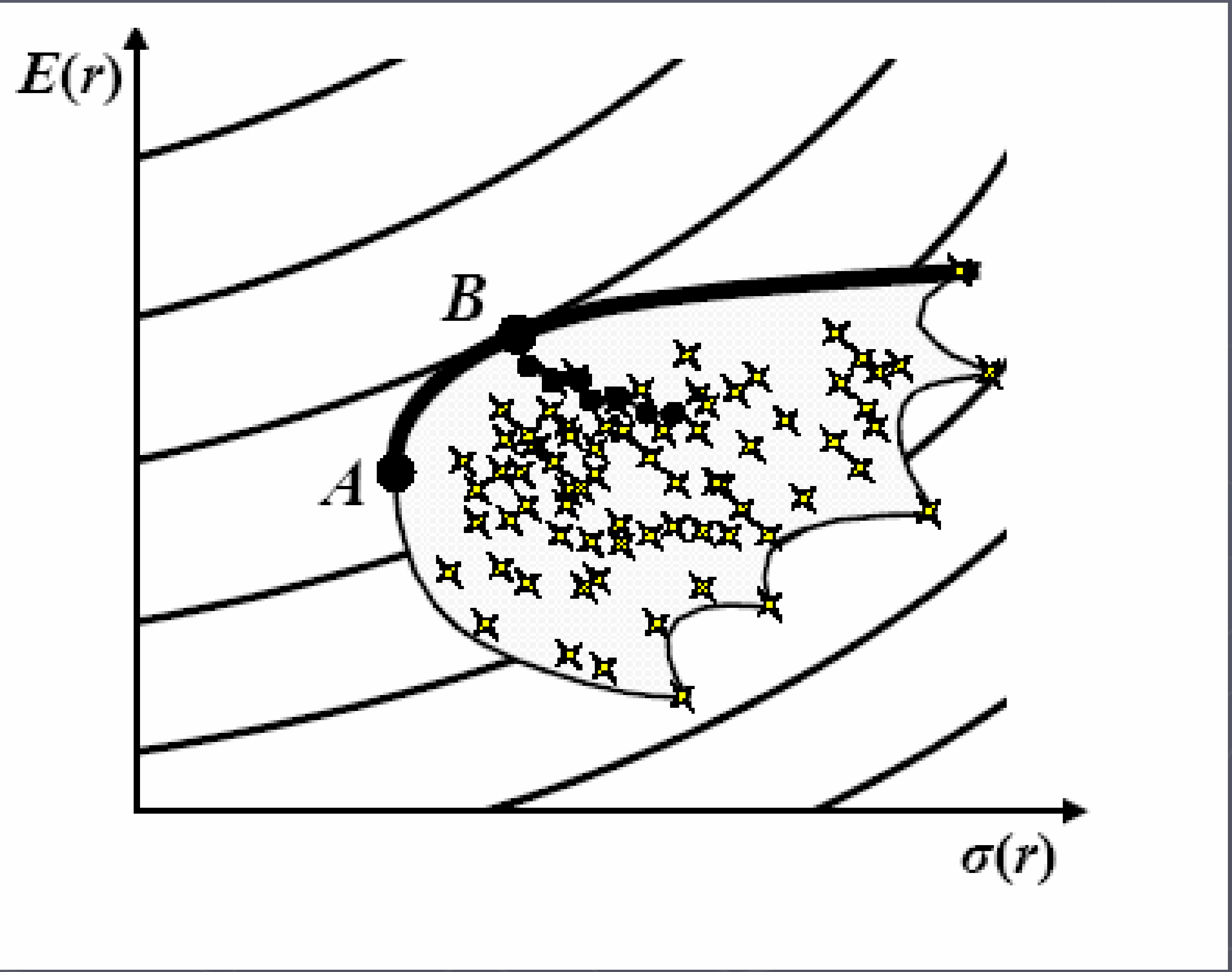
- **How to choose a portfolio?**

- Minimize risk of a given expected return? Or
- Maximize expected return for a given risk.

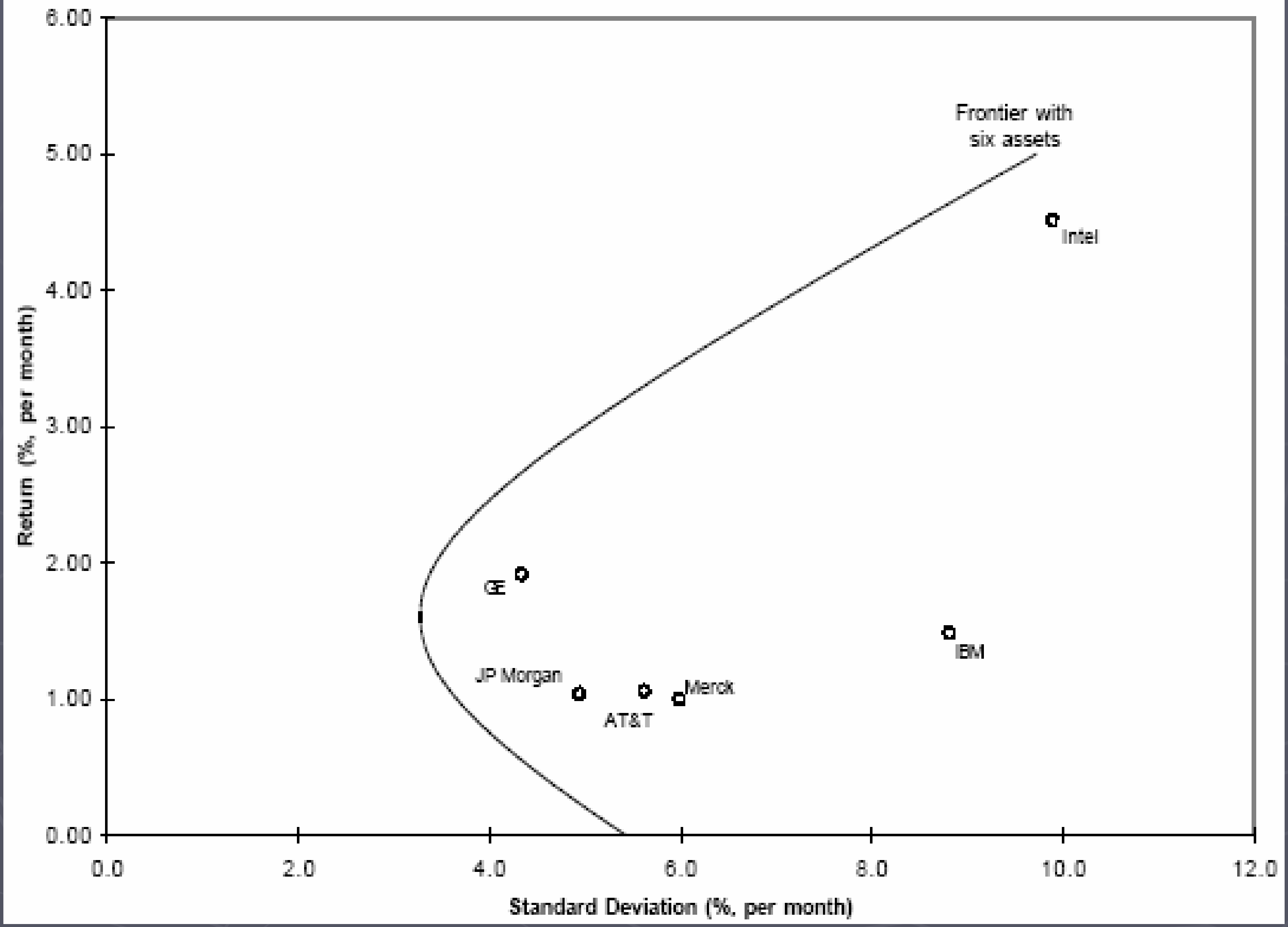
$$Minimize\ \sigma_v^2 = \sum_i^n \sum_j^n w_i w_j \sigma_{i,j} \quad subject\ to \quad (1) \sum_{i=1}^n w_i = 1$$

$$(2) \sum_{i=1}^n w_i r_i = \mu_v$$

Optimal portfolio selection (2)



Solving optimal portfolios "graphically"



Solving optimal portfolios

- The locus of all frontier portfolios in the plane is called ***portfolio frontier***
- The upper part of the portfolio frontier gives ***efficient frontier*** portfolios
- Minimal variance portfolio

Portfolio frontier with two assets

- Let $r_1 > r_2$ and let $w_1 = w$ and $w_2 = 1 - w$
- Then $\mu_v = wr_1 + (1-w)r_2$

$$\sigma_v^2 = w^2\sigma_1^2 + (1-w)\sigma_2^2 + 2w(1-w)\sigma_{1,2}$$

For a given μ_v there is a unique w that determines the portfolio with expected return

$$w = \frac{\mu_v - r_2}{r_1 - r_2}$$

Minimal variance portfolio

- We use $1 = uw^T$ and $\sigma_v^2 = wCw^T$
- Lagrange function $L(w,\lambda) = wCw^T - \lambda uw^T$

$$2wC - \lambda u = 0 \Rightarrow w = \frac{\lambda}{2} uC^{-1} \Big/ u^T$$

$$\Rightarrow 1 = \frac{\lambda}{2} uC^{-1}u^T \quad \Rightarrow \lambda = \frac{2}{uC^{-1}u^T} \quad \Rightarrow w = \frac{uC^{-1}}{uC^{-1}u^T}$$

Minimal variance curve

$$w = \frac{\left[mC^{-1}m^T - \mu_v uC^{-1}m^T \right] uC^{-1} + \left[\mu_v uC^{-1}u^T - mC^{-1}u^T \right] mC^{-1}}{uC^{-1}u^T mC^{-1}m^T - uC^{-1}m^T mC^{-1}u^T}$$

Where $mu_v = mw^T$

Some examples in MATLAB

$$\mu_1 = 0.2 \quad \sigma_1 = 0.25 \quad \rho_{1,2} = \rho_{2,1} = 0.3$$

$$\mu_2 = 0.13 \quad \sigma_2 = 0.28 \quad \rho_{2,3} = \rho_{3,2} = 0.0$$

$$\mu_3 = 0.17 \quad \sigma_3 = 0.20 \quad \rho_{1,3} = \rho_{3,1} = 0.15$$

$$m = \left[0.2\ 0.13\ 0.17\right]$$

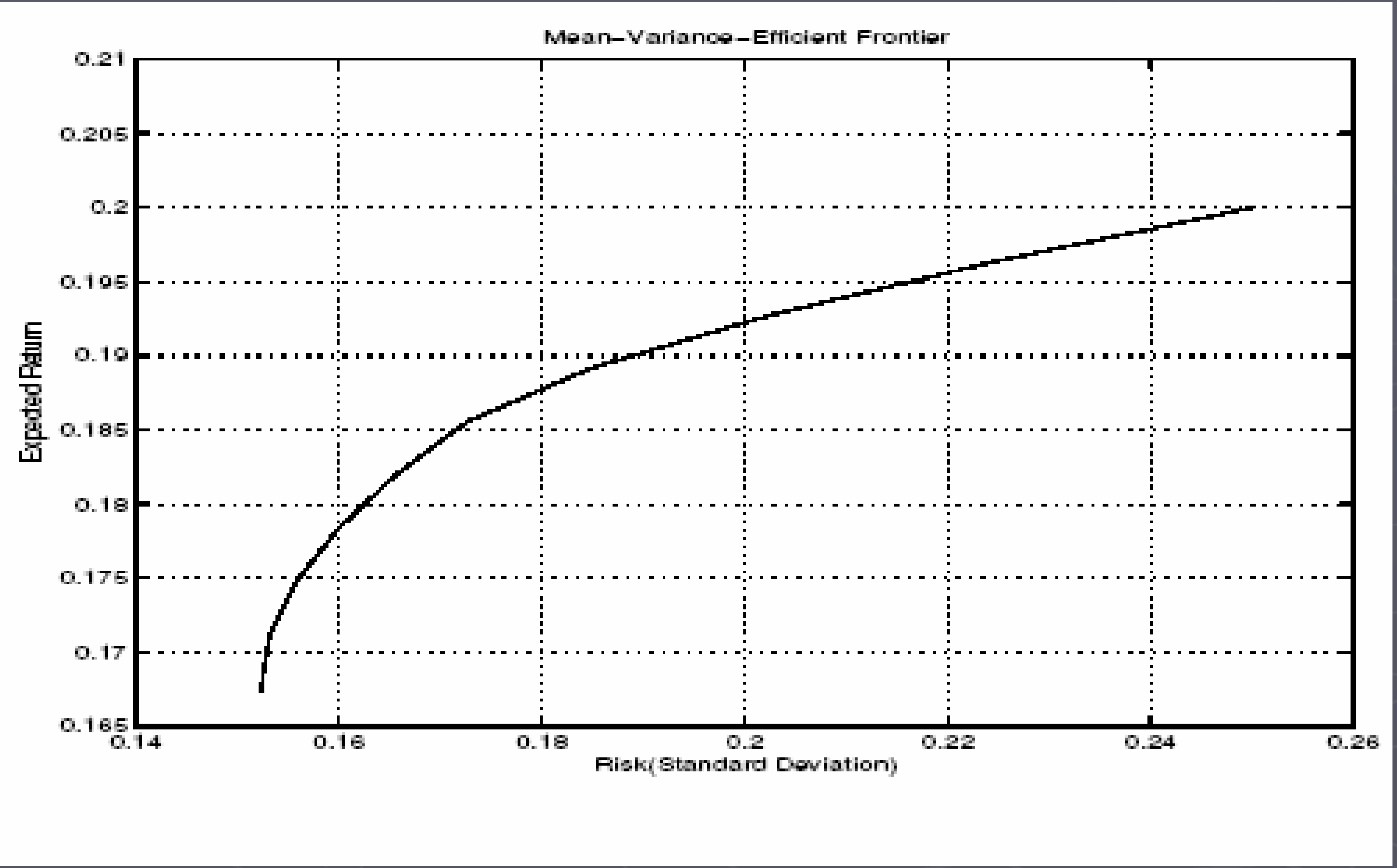
$$u = [1 \quad 1 \quad 1]$$

We calculate C and C^{-1}

$$w = \frac{uC^{-1}}{uC^{-1}u^T} = \left[-0.091 \quad 0.0679 \quad 1.0232\right]$$

$$w \cong \left[1.578 - 8.614\mu_v \quad 0.845 - 2.769\mu_v \quad -1.422 + 11.384\mu_v\right]$$

Using MATLAB



Examples in MATLAB(2)

- ***Frontcon*** function; with this function we can calculate some efficient portfolio

- [*pkock, preturn, pweigths*]= =**frontcon**(*returns,Cov,n,preturn,limits,group,grouplimits*)

Examples in MATLAB

- *pkock* – covariances of the returned portfolios
- *preturn* – returns of the returned portfolios
- *pweighs* – weighs of the returned portfolios
- *returns* – the stocks return
- *cov* – covariance matrix
- *n* – number of portfolios
- *group, group limits* – min and max weigh
- Other functions: ***portalloc, portopt***