BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

ADMISSION EXAM, July 21, 2019
Written test in MATHEMATICS

IMPORTANT NOTE:

1) The multiple choice problems in Part A may have one or more correct answers, which must be indicated by the candidate on the special form of the exam sheet. The multiple choice problems will be graded according to the partial scoring system of the exam regulation.

2) For the problems in Part B complete solutions are required, written on the exam sheet. They will be evaluated according to the corresponding grading table.

PART A

1. (6 points) For every $n \in \mathbb{N}^*$ denote

$$S_n = \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{(3n-2)\cdot(3n+1)}.$$

Then the value of S_{2019} is

A $\frac{2019}{6059}$; B $\frac{2018}{6059}$; C $\frac{2019}{6058}$; D $\frac{2018}{6058}$.

2. (6 points) If $\log_x(x^2+2x) + \log_{x^2}(x+2) = 4$, then x can be

A 2; B $\sqrt{2}$; C 4; D $2\sqrt{2}$.

3. (6 points) Let $A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & x \\ y & z \end{pmatrix}$ be matrices with real entries. If $AB = BA = O_2$ (the null matrix from $\mathcal{M}_2(\mathbb{R})$), then

A there is no such B; B B is uniquely determined;
C x, y, z are even numbers; D $\det B = 0$.

4. (6 points) In the ring $(\mathbb{Z}_{12}, +, \cdot)$ consider the equation $\hat{3}(x+\hat{2}) = \hat{9}$. Then

A the equation has exactly 4 solutions; B all its solutions are invertible in $(\mathbb{Z}_{12}, +, \cdot)$;
C the equation has exactly 3 solutions; D the equation has no solution.

5. (6 points) Let $\ell = \lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - x}{\sin^3 x}$. Which of the following statements are true?

A ℓ is a rational number. B The limit ℓ does not exist. C $\ell = 1/3$. D $\ell = \infty$.

6. (6 points) Let $a, b \in \mathbb{R}$ be parameters, let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} ae^x + b + e^{-x}, & \text{if } x \le 0 \\ x^2, & \text{if } x > 0 \end{cases}$$

and let $x_0 = 0$. Which of the following statements are true?

A There is an infinite number of pairs (a, b) for which f has limit at x_0.
B f is differentiable at $x_0 \Leftrightarrow (a = 1 \text{ and } b = -2)$.
C f is continuous at $x_0 \Leftrightarrow a + b = -1$.
D f is differentiable at $x_0 \Leftrightarrow (a = -2 \text{ and } b = 1)$.

7. (6 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = (x^2 - 4x + 6)e^x$. Then

$\boxed{\text{A}}$ the function f is decreasing on $(-\infty, 0]$ and increasing on $(0, +\infty)$;
$\boxed{\text{B}}$ 0 is an inflection point for the function f;
$\boxed{\text{C}}$ the function f is strictly increasing on $\mathbb{R}$;
$\boxed{\text{D}}$ the function f is convex on $\mathbb{R}$.

8. (6 points) The value of the integral $\displaystyle\int_0^{\pi/2} \frac{\mathrm{d}x}{1 + \sin x}$ is

$\boxed{\text{A}}$ $\frac{\pi}{2}$; $\boxed{\text{B}}$ $\frac{1}{2}$; $\boxed{\text{C}}$ π; $\boxed{\text{D}}$ 1.

9. (6 points) In the rhombus $ABCD$ we have $AB = 12$ and $m(\widehat{C}) = 60°$. Then the sum $\overrightarrow{AD} \cdot \overrightarrow{AB} + \overrightarrow{AC} \cdot \overrightarrow{BD}$ is equal to

$\boxed{\text{A}}$ 72; $\boxed{\text{B}}$ $72\sqrt{3}$; $\boxed{\text{C}}$ $144\sqrt{3}$; $\boxed{\text{D}}$ $72(1 + \sqrt{3})$.

10. (6 points) In the triangle ABC we have $BC = a$, $m(\widehat{A}) = 30°$ and $m(\widehat{B}) = 105°$. Then the area of the triangle ABC is equal to

$\boxed{\text{A}}$ $\dfrac{a^2(\sqrt{3} - 1)}{4}$; $\boxed{\text{B}}$ $\dfrac{a^2(1 + \sqrt{3})}{4}$; $\boxed{\text{C}}$ $\dfrac{a^2(\sqrt{3} + \sqrt{2})}{4}$; $\boxed{\text{D}}$ $\dfrac{a^2\sqrt{3}}{4}$.

PART B

1. (10 points) Compute the following limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{\sqrt{n^2 + k^2}}$; (b) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{\sqrt{n^2 + k}}$.

2. (10 points) In the cartesian system xOy consider the points $A(2, -1)$, $B(4, 3)$ and the line $d : x - 2y - 1 = 0$.

(a) Prove that the line d passes through the middle of the segment $[AB]$.

(b) Determine the points C for which the area of the triangle ABC is equal to 3 and one of the medians of the triangle ABC lies on the line d.

3. (10 points) Consider the set of matrices

$$G = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \middle| z_1, z_2 \in \mathbb{C} \right\} \subseteq \mathcal{M}_2(\mathbb{C}),$$

where by $\overline{z}$ we denote the conjugate of a complex number z.

(a) Show that G is a subgroup of the group $(\mathcal{M}_2(\mathbb{C}), +)$.

(b) Construct an injective group homomorphism between the groups $(\mathbb{C}, +)$ and $(G, +)$.

NOTE:
All subjects are compulsory. 10 points are given by default.
The work time is 3 hours.

Answers and solutions

PART A

Answers to problems:

1. $\boxed{\text{C}}$; **2.** $\boxed{\text{A}}$; **3.** $\boxed{\text{B}}$, $\boxed{\text{C}}$, $\boxed{\text{D}}$; **4.** $\boxed{\text{C}}$; **5.** $\boxed{\text{A}}$, $\boxed{\text{C}}$;

6. $\boxed{\text{A}}$, $\boxed{\text{B}}$, $\boxed{\text{C}}$; **7.** $\boxed{\text{C}}$, $\boxed{\text{D}}$; **8.** $\boxed{\text{D}}$; **9.** $\boxed{\text{A}}$; **10.** $\boxed{\text{B}}$.

Solutions to problems:

1. For every $k \in \mathbb{N}^*$ we have the equality $\frac{1}{k(k+3)} = \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right)$, whence it follows that for every $n \in \mathbb{N}^*$ we have

$$3S_n = \frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \cdots + \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{3n+1},$$

that is, $S_n = \frac{n}{3n+1}$ for every $n \in \mathbb{N}^*$. Hence $S_{2019} = \frac{2019}{6058}$.

2. The existence conditions for the logarithms lead to $x \in (0,1) \cup (1, \infty)$. Then the equation is equivalent to

$$\frac{1}{2}\log_x(x+2) + 1 + \log_x(x+2) = 4,$$

whence it follows that $\log_x(x+2) = 2$, that is, $x^2 = x + 2$. In conclusion, $x = 2$ is the unique solution.

3. From the matrix equations $AB = BA = O_2$ we get

$$\begin{pmatrix} 4-2y & 4x-2z \\ -2+y & -2x+z \end{pmatrix} = \begin{pmatrix} 4-2x & -2+x \\ 4y-2z & -2y+z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By solving the resulting systems of equations we conclude that the unique solution is $x = 2$, $y = 2$ and $z = 4$, hence $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

4. The equation can be rewritten equivalently as follows: $\hat{3}(x + \hat{2}) = \hat{9} \Leftrightarrow \hat{3}x + \hat{6} = \hat{9} \Leftrightarrow \hat{3}x - \hat{3} = \hat{0} \Leftrightarrow \hat{3}(x - \hat{1}) = \hat{0}$. The latter equality holds if and only if $x - \hat{1} \in \{\hat{0}, \hat{4}, \hat{8}\}$, hence the set of solutions is $M = \{\hat{1}, \hat{5}, \hat{9}\}$. Out of them the solution $\hat{9}$ is not invertible in the ring $(\mathbb{Z}_{12}, +, \cdot)$.

5. By using l'Hôpital's rule, we get

$$\ell = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{3\sin^2 x \cos x} = \lim_{x\to 0} \frac{1 - \cos^2 x}{3\sin^2 x \cos^3 x} = \lim_{x\to 0} \frac{1}{3\cos^3 x} = \frac{1}{3}.$$

6. We have $\lim\limits_{x\nearrow 0} f(x) = a + b + 1 = f(0)$ and $\lim\limits_{x\searrow 0} f(x) = 0$. Therefore, f is continuous at $x_0 = 0$ if and only if $a + b = -1$, hence there is an infinite number of pairs (a, b) for which f is continuous at x_0. Also, f is differentiable on $\mathbb{R} \setminus \{0\}$, and

$$f'(x) = ae^x - e^{-x} \quad \text{for every } x < 0,$$

$$f'(x) = 2x \quad \text{for every } x > 0.$$

It is easily seen that $\lim\limits_{x\nearrow 0} f'(x) = a - 1$ and $\lim\limits_{x\searrow 0} f'(x) = 0$. Assuming that $a + b = -1$ (that is, f is continuous at 0) and using a consequence of Lagrange's mean value theorem, it follows that

$f_s'(0) = a - 1$ and $f_d'(0) = 0$. Therefore, f is differentiable at $x_0 = 0$ if and only if $a + b = -1$ and $a - 1 = 0$, that is, if and only if $a = 1$ and $b = -2$.

7. We have $f'(x) = (x^2 - 2x + 2)e^x > 0$, for every $x \in \mathbb{R}$, and $f''(x) = x^2 e^x \geq 0$, for every $x \in \mathbb{R}$.

8. With the change of variable $\operatorname{tg} \frac{x}{2} = t$, we get

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{1 + \sin x} = \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2}} \cdot \frac{2}{1 + t^2}\,\mathrm{d}t = 2\int_0^1 \frac{\mathrm{d}t}{(1+t)^2} = -\frac{2}{1+t}\bigg|_0^1 = 1.$$

9. In the rhombus $ABCD$ we have $m(\widehat{BAD}) = 60^\circ$ and the diagonals are perpendicular. Using the definition of the scalar product and denoting by O the intersection of the diagonals, we get

$$\overrightarrow{AD} \cdot \overrightarrow{AB} + \overrightarrow{AC} \cdot \overrightarrow{BD} = AD \cdot AB \cdot \cos(\widehat{BAD}) + AC \cdot BD \cdot \cos(\widehat{AOD}) = 12 \cdot 12 \cdot \frac{1}{2} + 0 = 72.$$

10. We have $m(\widehat{C}) = 180^\circ - 30^\circ - 105^\circ = 45^\circ$. We use the law of sines in the triangle ABC:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Leftrightarrow \frac{a}{\frac{1}{2}} = \frac{c}{\frac{\sqrt{2}}{2}} \Leftrightarrow c = a\sqrt{2}.$$

The area of the triangle ABC is

$$\mathcal{A}(ABC) = \frac{a \cdot c \cdot \sin B}{2} = \frac{1}{2}a^2\sqrt{2} \cdot \sin(60^\circ + 45^\circ) = \frac{1}{2}a^2\sqrt{2} \cdot \left(\frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}\right) = \frac{a^2(1+\sqrt{3})}{4}.$$

PART B

Solutions to problems and grading table:

1. **(a)** Let $a_n = \frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sqrt{n^2+k^2}}$.

(**3 points**) We have

$$a_n = \frac{1}{n}\sum_{k=1}^{n} \frac{k/n}{\sqrt{1+(k/n)^2}} = \sigma(f, \Delta_n, \xi_n),$$

where $f : [0,1] \to \mathbb{R}$ is the function defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$, Δ_n is the division of the interval $[0,1]$ defined by $\Delta_n = (0,\, 1/n,\, 2/n,\, \ldots,\, 1)$, and $\xi_n = (1/n,\, 2/n,\, \ldots,\, 1) \in P(\Delta_n)$.

Remark. If the function, the division and the intermediate points system are not specified, then there will awarded 2 points out of the 3 points.

(**1 point**) Since $\|\Delta_n\| = 1/n \to 0$ when $n \to \infty$, it follows that $\lim\limits_{n\to\infty} a_n = \int_0^1 f(x)\,\mathrm{d}x$.

Remark. If it is not mentioned that $\|\Delta_n\| = 1/n \to 0$ when $n \to \infty$, then the point is not awarded.

(**3 points**) Therefore, $\lim\limits_{n\to\infty} a_n = \int_0^1 \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = \sqrt{1+x^2}\,\Big|_0^1 = \sqrt{2}-1$.

(b) Let $b_n = \frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sqrt{n^2+k}}$. We have

$$(\textbf{1 point}) \qquad b_n \le \frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sqrt{n^2+1}} = \frac{1}{n\sqrt{n^2+1}}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n\sqrt{n^2+1}}$$

and analogously

$$(\textbf{1 point}) \qquad b_n \ge \frac{n(n+1)}{2n\sqrt{n^2+n}}.$$

(**1 point**) As $\lim\limits_{n\to\infty} \frac{n(n+1)}{2n\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \frac{n(n+1)}{2n\sqrt{n^2+n}} = \frac{1}{2}$, using the squeeze theorem it follows that $\lim\limits_{n\to\infty} b_n = 1/2$.

2. **(a)** (**1 point**) The middle of the segment $[AB]$ is the point $M(3,1)$.
(**1 point**) The coordinates of the point M satisfy the equation of the line $d:\ 3-2-1=0$.

(b) (**2 points**) Since the line d contains a median of the triangle ABC, and the coordinates of the points A and B do not satisfy the equation of the line d (or since d passes through the middle of the segment $[AB]$), we get that the point C belongs to the line d.
(**2 points**) Consider the point $C(c_1, c_2)$. Then we have $c_1 - 2c_2 - 1 = 0 \Leftrightarrow c_1 = 2c_2 + 1$.
(**2 points**) The area of the triangle ABC is $\frac{1}{2}|\Delta| = 3$, where

$$\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 3 & 1 \\ 2c_2+1 & c_2 & 1 \end{vmatrix}.$$

(**1 point**) We get $|-6c_2+6| = 6 \Leftrightarrow |-c_2+1| = 1$, and so $c_2 = 0$ or $c_2 = 2$.
(**1 point**) Therefore, the coordinates of C can be $C(1,0)$ or $C(5,2)$.

3. (a) (1 point) We have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in G$, hence $G \neq \emptyset$.

Let $A = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}$, $B = \begin{pmatrix} z_3 & z_4 \\ -\overline{z_4} & \overline{z_3} \end{pmatrix} \in G$. We have:

$$\textbf{(3 points)} \quad A + B = \begin{pmatrix} z_1 + z_3 & z_2 + z_4 \\ -\overline{z_2} - \overline{z_4} & \overline{z_1} + \overline{z_3} \end{pmatrix} = \begin{pmatrix} z_1 + z_3 & z_2 + z_4 \\ -\overline{z_2 + z_4} & \overline{z_1 + z_3} \end{pmatrix} \in G,$$

$$\textbf{(3 points)} \quad -A = \begin{pmatrix} -z_1 & -z_2 \\ \overline{z_2} & -\overline{z_1} \end{pmatrix} = \begin{pmatrix} -z_1 & -z_2 \\ -\overline{(-z_2)} & \overline{-z_1} \end{pmatrix} \in G.$$

Hence G is a subgroup of the group $(\mathcal{M}_2(\mathbb{C}), +)$.

Remark. Alternatively, the set G is a subgroup of the group $(\mathcal{M}_2(\mathbb{C}), +) \Leftrightarrow [G \neq \emptyset$, $A - B \in G$ for every $A, B \in G] \Leftrightarrow [G$ is a stable subset of $(\mathcal{M}_2(\mathbb{C}), +)$ and $(G, +)$ is a group$]$.

(b) (1 point) Let $f : \mathbb{C} \to G$ be the function defined by $f(z) = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$ for every $z \in \mathbb{C}$. Notice that $\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \in G$ for every $z \in \mathbb{C}$, hence f is well defined.

(1 point) Checking the injectivity of f.

(1 point) For every $z_1, z_2 \in \mathbb{C}$ we have:

$$f(z_1 + z_2) = \begin{pmatrix} z_1 + z_2 & 0 \\ 0 & \overline{z_1 + z_2} \end{pmatrix} = \begin{pmatrix} z_1 + z_2 & 0 \\ 0 & \overline{z_1} + \overline{z_2} \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & \overline{z_1} \end{pmatrix} + \begin{pmatrix} z_2 & 0 \\ 0 & \overline{z_2} \end{pmatrix} = f(z_1) + f(z_2).$$

Hence f is a group homomorphism between the groups $(\mathbb{C}, +)$ and $(G, +)$.

Remark. There are other injective group homomorphisms between the groups $(\mathbb{C}, +)$ and $(G, +)$, which can be defined, for instance, by $f(z) = \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}$ or $f(z) = \begin{pmatrix} z & z \\ -\overline{z} & \overline{z} \end{pmatrix}$.