BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

ADMISSIONS EXAM, September 12th 2018
Written test in MATHEMATICS

IMPORTANT NOTE:

1) Multiple choice questions (Part A) may have one or several correct answers. Candidates must identify all correct answers and write them on the exam sheet. To get full credit, all correct answers must be identified and only those.

2) Problems in Part B must be solved entirely on the exam sheet. These will be evaluated in details according to the solution key.

PART A

1. (5 points) The intersection of the Ox axis and the graph of the function

$$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases} 2^{x^2}-4, & \text{if } x<2\\ x-2, & \text{if } x\geq 2\end{cases}$$

$\boxed{\text{A}}$ is the empty set; $\boxed{\text{B}}$ is a set containing one element; $\boxed{\text{C}}$ is a set containing two elements; $\boxed{\text{D}}$ is a set containing three elements; $\boxed{\text{E}}$ is a set containing four elements.

2. (5 points) Consider the triplet $(\mathbb{R},+,\cdot)$, where $\mathbb{R}$ is the set of real numbers and $+,\ \cdot$ are the usual addition and multiplication operations. Determine which of the following statements are true:

$\boxed{\text{A}}$ $(\mathbb{R},+,\cdot)$ is a group; $\boxed{\text{B}}$ $(\mathbb{R},+,\cdot)$ is a ring; $\boxed{\text{C}}$ $(\mathbb{R},+,\cdot)$ is a field; $\boxed{\text{D}}$ $(\mathbb{R},+,\cdot)$ is a ring, but not a field; $\boxed{\text{E}}$ $(\mathbb{R},+,\cdot)$ is not a ring.

3. (5 points) The value of the limit $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin^2 x)}{x\cdot \operatorname{tg}(2x)}$ is

$\boxed{\text{A}}$ $-\infty$; $\boxed{\text{B}}$ $\frac{1}{2}$; $\boxed{\text{C}}$ 2; $\boxed{\text{D}}$ $+\infty$; $\boxed{\text{E}}$ 0.

4. (5 points) In a cartesian coordinate system xOy, consider an equilateral triangle OAB with side ℓ, such that the vertex A is on the line with equation $x-\sqrt{3}y=0$. The coordinates of vertex B can be:

$\boxed{\text{A}}$ $(0,\ell)$; $\boxed{\text{B}}$ $\left(\frac{\ell\sqrt{3}}{2},-\frac{\ell}{2}\right)$; $\boxed{\text{C}}$ $(0,-\ell)$; $\boxed{\text{D}}$ $\left(-\frac{\ell\sqrt{3}}{2},\frac{\ell}{2}\right)$; $\boxed{\text{E}}$ $\left(-\frac{\ell\sqrt{3}}{2},-\frac{\ell}{2}\right)$.

5. (5 points) If $\cos\beta$ is the geometric mean of the numbers $\sin\alpha$ and $\cos\alpha$, where $0<\alpha,\beta<\frac{\pi}{2}$, then $\cos 2\beta$ is equal to:

$\boxed{\text{A}}$ $-2\sin^2\left(\frac{\pi}{4}-\alpha\right)$; $\boxed{\text{B}}$ $-2\cos^2\left(\frac{\pi}{4}+\alpha\right)$; $\boxed{\text{C}}$ $2\sin^2\left(\frac{\pi}{4}+\alpha\right)$; $\boxed{\text{D}}$ $2\cos^2\left(\frac{\pi}{4}-\alpha\right)$; $\boxed{\text{E}}$ $2\sin^2\left(\frac{\pi}{4}-\alpha\right)$.

6. (5 points) Consider the function $f:\mathbb{R}^*\to\mathbb{R}$, given by

$$f(x)=x^2+\frac{2}{x},\quad \forall x\in\mathbb{R}^*.$$

Which of the following statements are true?

$\boxed{\text{A}}$ f does not have an oblique asymptote; $\boxed{\text{B}}$ f has one single point of local minimum; $\boxed{\text{C}}$ equation $f(x)=3$ has solution $x=1$; $\boxed{\text{D}}$ f is monotone on $\mathbb{R}^*$; $\boxed{\text{E}}$ f is convex on $\mathbb{R}^*$.

PART B

1. a) (8 points) Find all the pairs of real numbers with the property that both their sum and their product are equal to -2.

b) (12 points) Determine the rank of the matrix $A = \begin{pmatrix} a & a & a & 1 \\ a & a^2 & 4 & 1 \\ a & -2 & -2 & 1 \end{pmatrix}$. Discuss all the possibilities according to the values of the real parameter a.

2. a) (8 points) Let P and Q be the midpoints of the diagonals $[AC]$ and $[BD]$ of the convex quadrilateral $ABCD$.

1. Prove that $\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{AD} - \overrightarrow{BC}\right)$.
2. Show that if $\overrightarrow{AD} + \overrightarrow{CB} = 3\overrightarrow{PQ}$, then the quadrilateral $ABCD$ is a parallelogram.

b) (7 points) Prove the equality:

$$\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4.$$

3. Let a, b be real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 + ax + b, & x \leq 0 \\ x - 1, & x > 0 \end{cases}.$$

a) (10 points) Find the numbers a, b for which f is differentiable and f' is continuous.
b) (8 points) Find the primitives of the function f when they exist.
c) (7 points) Compute the integral $\int_{-1}^{1} f(x)\, dx$ for $a = 1$ and $b = -1$.

NOTE:
All subjects are mandatory. Grading starts at 10 points.
Working time is 3 hours.