Symmetrization methods in the study of sublinear elliptic problems

Summary

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Introduction

In the last few years, many mathematicians dealt with the study of such phenomenon, in the theory of partial differential equations, which involve symmetrizations. Certainly we can not list all of these works, but never the less we would like to mention the work of J. V. Schaftingen [64] and S. Kesenvan [38], F. Faraci et al. [21]. These results connect the theory of calculus of variations with results from the theory of symmetrizations. Thus, we can see that such studies rely on two main pillars: calculus of variations and symmetrizations.

The aim of the thesis is to present new existence and the multiplicity results in the study of sublinear elliptic problems in different contexts, combining techniques from calculus of variations and also from the theory of symmetrizations. The whole work is based on the articles [23, 25–28].

The calculus of variations continues to be an area of a very rapid growth. Variational methods are indispensable as a tool in mathematical physics and geometry. Calculus of variations has broad applications in other fields of mathematics and in many areas of physics (e.g. calculating trajectories and geodesics in both classical mechanics and general relativity), aeronautics (e.g. maximizing the lift of an air plane wing), sporting equipment design (e.g. minimizing air resistance on a bicycle helmet, optimizing the shape of a ski), mechanical engineering (e.g maximizing the strength of a column, a dam, or an arch), boat design (e.g optimizing the shape of a boat hull), and it has been introduced recently in economics, biology, etc. Here we would like to highlight the recent books of A. Kristály, V. D. Rădulescu, Cs. Varga [43], and the books of H. Brézis [10] and M. Struwe [71].

Despite the fact, that symmetrizations don't really occur in modelling real situations of the everyday life, they are very useful and are highly applied topic in the theory of partial differential equations. Many mathematicians worked and work in the field of symmetrizations, trying to describe new phenomena. Here we mention the works of S. Kesevan [38], Brock and Solynin [11], J. Van Schaftingen [64], M. Squassina [70] who have proven many results among symmetrizations in the past few years, here we are thinking of the symmetric minimax principle, Ekeland-, Borwein-Preiss variational principles etc., which have opened many new ways for applications of this topic. In [11], Brock and Solynin proved that the Steiner symmetrization of a function can be approximated in $L^p(\mathbb{R}^n)$ by a sequence of very simple rearrangements which are called polarizations. Moreover, they introduced the concept of rearrangement and investigated some general properties.

In the sequel we sketch the structure of the thesis.

In the first part of the thesis we introduce the basic definitions and results from the theory of Lebesgue spaces and Sobolev spaces, from the theory of calculus of variations, from the theory of locally Lipschitz functions, and finally from Finsler geometry.

In the second part, we present some new existence and multiplicity results. To be more precise, in the first part of Chapter 6, we will study the following quasilinear equation coupled with a homogeneous Neumann boundary condition:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda\alpha(x,y)f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{N}_\lambda$}$$

where $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being bounded and open with smooth boundary, $p > N$, $N - m \geq 2$, Δ_p is the p-Laplacian operator, λ is a positive parameter, $\alpha \in L^\infty(\Omega)$ is a non-zero potential with compact support, n is the outward normal vector, and $f : [0, \infty[\to \mathbb{R}$ is a continuous function with $f(0) = 0$. First we establish a compact embedding result (see Theorem 6.3), then we prove a multiplicity result for the problem $(\mathcal{N}_\lambda)$ on strip-like domains. Using variational methods, we prove that for large values of λ, problem $(\mathcal{N}_\lambda)$ has at least two non-zero weak solutions, while there exists at least a $\widetilde{\lambda} > 0$ such that problem $(\mathcal{N}_{\widetilde{\lambda}})$ has at least three non-zero weak solutions. These solutions show symmetry properties with respect to certain group actions (see Theorem 6.1).

In this chapter we also want to apply the same method which was presented above, for the following quasilinear equation coupled with a Neumann boundary condition (see Chapter 6, Section 6.2)

$$\begin{cases} -\Delta_p u + |u|^{p-2} \cdot u = \beta(x)g(u) & \text{in } \quad \Omega \\ \frac{\partial u}{\partial n} = \lambda\alpha(x)f(u) & \text{on } \quad \partial\Omega, \end{cases} \tag{$\mathcal{N}_\lambda$}$$

here again $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being with smooth boundary, $p > N$, $\alpha \in L^\infty(\partial\Omega) \cap L^1(\partial\Omega), \beta \in L^1(\Omega)$ are non-zero positive potentials, and

$f, g : [0, +\infty[\to \mathbb{R}$ are continuous functions with $f(0) = g(0) = 0$. We prove that for large values of λ, problem $(\mathcal{N}_\lambda)$ has at least two non-zero symmetric invariant weak solutions.

In the last part of Chapter 6, we also consider the following sublinear elliptic differential inclusion problem coupled with a zero Dirichlet boundary condition (see Chapter 6, Section 6.3):

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda\alpha(x,y)\partial F(u(x,y)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being bounded and open with smooth boundary, $p > N$, $N - m \geq 2$, λ is a positive parameter, $\alpha \in L^\infty(\Omega) \cap L^1(\Omega)$ is a axially symmetric, non-negative, non-zero function and ∂F stands for the generalized gradient of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. We prove that for large values of λ, problem $(\mathcal{P}_\lambda)$ has at least two non-zero cylindrically symmetric weak solutions.

In Chapter 7 we consider the following quasi-linear, elliptic differential system coupled with a homogeneous Dirichlet boundary condition,

$$\begin{cases} -\Delta_p u = \lambda F_u(x, u(x), v(x)) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x, u(x), v(x)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{S}_\lambda$}$$

where λ is a positive parameter and $N > p, q > 1$, $\Omega = B(0,1) \subset \mathbb{R}^N$ is the unit ball, $F \in \mathcal{C}^1(\Omega \times \mathbb{R}^2, \mathbb{R})$, F_z denotes the partial derivative of F with respect to z. We prove that for large values of λ, problem $(\mathcal{S}_\lambda)$ has at least two non-zero symmetric invariant (spherical cap symmetric invariant) weak solutions.

In the last Chapter, we establish uniqueness, location and rigidity results for the (singular) Poisson equation involving the Finsler-Laplace operator on Finsler-Hadamard manifolds having finite reversibility constant, i.e., we will consider the following problem:

$$\begin{cases} \boldsymbol{\Delta}(-u) - \mu \frac{u}{d_F^2(x_0, x)} = 1 & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}_\Omega^\mu$}$$

where $\boldsymbol{\Delta}$ denotes the Finsler-Laplace operator on (M, F), d_F is the metric function, $x_0 \in \Omega$ is fixed, $\mu \geq 0$ is a parameter, and $\Omega \subset M$ is an open and bounded domain with sufficiently smooth boundary.

We specify here our original results:

Chapter 6: Theorem 6.1, Remark 6.2, Theorem 6.3, Proposition 6.4, Corollary 6.5, Example 6.6, Theorem 6.7, Example 6.8, Lemma 6.10, Lemma 6.11, Lemma 6.12, Lemma 6.13, Lemma 6.14, Lemma 6.15, Theorem 6.16, Remark 6.17, Proposition 6.18, Proposition 6.19.

Chapter 7: Theorem 7.1, Remark 7.2, Remark 7.3, Remark 7.4, Lemma 7.6, Lemma 7.7, Lemma 7.9.

Chapter 8: Theorem 8.1, Theorem 8.2, Proposition 8.3, Theorem 8.4, Remark 8.5, Proposition 8.6, Remark 8.7, Theorem 8.8, Proposition 8.9, Theorem 8.10, Proposition 8.11, Remark 8.12, Proposition 8.13, Theorem 8.14.

Finally, we mention five other papers which also contain original results, but these results are not included in this thesis, since these results could not be bounded directly to the topic of the thesis, and they would have destructed the unity of it.

- CS. FARKAS, *A generalized form of Ekeland's variational principle*, Analele Stiintifice ale Universitatii Ovidius Constanta, Vol. 20 (1), pages 101-112, 2012.
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- F. FARACI, CS. FARKAS, *New conditions for the existence of infinitely many solutions for a quasilinear problem*, accepted, Proceedings of Edinburgh Mathematical Society, 2014.
- F. FARACI, CS. FARKAS, *A quasilinear elliptic problem involving critical Sobolev exponents*, accepted, Collectanea Mathematica, 2014.

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Csaba Farkas
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Part I

Preliminaries

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L^p and Sobolev spaces

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Elements from the theory of locally Lipschitz functions

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Symmetrizations

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Part II

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Group-invariant multiple solutions on strip-like domains

In this chapter we present some multiplicity results on strip-like domains, depending on a positive parameter λ. These solutions show symmetry properties with respect to certain group actions.

Contents

In this chapter we present some multiplicity results on strip like domains. This chapter is based on the articles [23, 25, 28].

6.1 Group-invariant multiple solutions

6.1.1 Formulation of the problem

Let us consider the following quasilinear equation coupled with a homogeneous Neumann boundary condition:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda \alpha(x,y) f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{N}_\lambda$}$$

where $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being bounded and open with smooth boundary, $p > N$, $N - m \geq 2$, Δ_p is the p-Laplacian operator, λ is a positive parameter, $\alpha \in L^\infty(\Omega)$ is a non-zero potential with compact support, n is the outward normal vector and $f : [0, \infty[\to \mathbb{R}$ is a continuous function with $f(0) = 0$.

The purpose of this section is to ensure the existence of multiple solutions for the problem $(\mathcal{N}_\lambda)$, where the natural function space is $W^{1,p}(\Omega)$. With respect to [20], the main difficulty to treat this problem comes from the unboundedness of the domain, i.e. $\Omega = \omega \times \mathbb{R}^{N-m}$; indeed, no compact embedding is available for $W^{1,p}(\Omega)$ into any Lebesgue space. Since $p > N$, due to Morrey's theorem (see [10]), the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is continuous, without being compact. The latter fact is due to the lack of concentration compactness along certain directions inside of the strip-like domain $\Omega = \omega \times \mathbb{R}^{N-m}$. On the other hand, Faraci, Iannizzotto and Kristály (see [21]) proved a compact embedding result for cylindrically symmetric functions on strip-like domains in low dimensions; namely, if $p > N$, the subspace of cylindrically symmetric functions of $W^{1,p}(\Omega)$, denoted in the sequel by $W^{1,p}_c(\Omega)$ is compactly embedded into $L^\infty(\Omega)$. More precisely, the (closed) subspace of $W^{1,p}(\Omega)$ consisting of the *cylindrically symmetric* functions is defined by

$$W_c^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u(x,\,\cdot\,) \text{ is radially symmetric for all } x \in \omega\}. \tag{6.1.1}$$

It is clear that such a compactness result, combined with a suitable variational method, figures out to be a hopeful argument to handle problem $(\mathcal{N}_\lambda)$. Indeed, in [21], by assuming suitable oscillatory behavior on the nonlinear term f (at zero and at infinity), the authors guaranteed the existence of a whole sequence of weak solutions for problem $(\mathcal{N}_\lambda)$, by exploiting the aforementioned compactness result, the principle of symmetric criticality and a recent variational principle of Ricceri.

As we mentioned above the purpose of the present section is to guarantee two or three non-zero weak solutions to problem $(\mathcal{N}_\lambda)$ which are invariant with respect to certain groups, whenever the nonlinear term verifies unusual growth assumptions, similar to the ones considered in the paper [20] (for a Dirichlet problem studied on bounded domains). In order to present our main result, we first recall that $u \in W^{1,p}(\Omega)$ is a *weak solution* of the problem $(\mathcal{N}_\lambda)$ if for all $v \in W^{1,p}(\Omega)$, we have

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dxdy = \lambda \int_\Omega \alpha(x,y)f(u)vdxdy. \tag{6.1.2}$$

Assume that $N - m \geq 2$, and let

$$\mathcal{G}_{N,m} = \{G = \mathrm{id}_{\mathbb{R}^m} \times O(k_1) \times ... \times O(k_l) : k_1, ..., k_l \geq 2,\; k_1 + ... + k_l = N - m\}.$$

Now, let $G \in \mathcal{G}_{N,m}$ be fixed. We introduce the action of the group G on the Sobolev space $W^{1,p}(\Omega)$ by

$$\tilde{g}u(x,y) = u(x, g^{-1}y) \text{ for all } \tilde{g} = \mathrm{id}_{\mathbb{R}^m} \times g \in G,\; (x,y) \in \Omega,\; u \in W^{1,p}(\Omega).$$

Note that G acts linearly, continuously and isometrically on $W^{1,p}(\Omega)$, i.e., $\|\tilde{g}u\| = \|u\|$ for every, $\tilde{g} \in G$ and $u \in W^{1,p}(\Omega)$. Let

$$W_G^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \tilde{g}u = u \text{ for all } \tilde{g} \in G\} \tag{6.1.3}$$

be the subspace of $G-$invariant functions in $W^{1,p}(\Omega)$. In particular, one has that

$$W^{1,p}_{\mathrm{id}_{\mathbb{R}^m} \times O(N-m)}(\Omega) = W_c^{1,p}(\Omega).$$

We say that a function $h : \Omega \to \mathbb{R}$ is *$G-$invariant* if $h(\tilde{g}(x,y)) = h(x,y)$, for all $\tilde{g} = \mathrm{id}_{\mathbb{R}^m} \times g \in G$ and $(x,y) \in \Omega$. The function h is called *cylindrically symmetric*, if it is $\mathrm{id}_{\mathbb{R}^m} \times O(N-m)-$invariant.

For later use, if K is a subset of $\mathbb{R}^N$ and $\mu > 0$, we introduce the notation

$$K_\mu = \{x \in \Omega : \mathrm{dist}(x, K) \le \mu\},$$

$|A|$ is the Lebesgue measure of the set A, while $c_\infty > 0$ is the embedding constant of the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Based on [10], after elementary estimations, one certainly has $c_\infty \le \frac{2p}{p-N}$. Now, we are in the position to state the main result of this section.

Theorem 6.1 *Let $p > N \ge m+2$, $\Omega = \omega \times \mathbb{R}^{N-m}$, $\alpha \in L^\infty(\Omega)$ be a cylindrically symmetric, non-negative, non-zero function with compact support K. Let $\mu = \mathrm{dist}(K, \partial\Omega)$. Let $f : [0,\infty[\to \mathbb{R}$ be a continuous function with $f(0) = 0$ and assume that*

i) $M_F = \sup\limits_{[0,\infty[} F < \infty$, *where* $F(s) = \int_0^s f(t)dt$;

ii) $\lim\limits_{s\to 0^+} \dfrac{f(s)}{s^{p-1}} = 0.$

Moreover, assume that there exists $r > 0$ such that

iii) $F(r) = \max\limits_{[0, c_\infty (p M_F \|\alpha\|_{L^1})^{\frac{1}{p}}]} F < M_F$;

iv) $\dfrac{F(r)}{r^p} > \dfrac{1}{p\|\alpha\|_1} \left[\dfrac{|K_\mu \setminus K|}{\mu^p} + |K_\mu|\right].$

Then the following statements hold:

(a) *For every $G \in \mathcal{G}_{N,m}$ and $\lambda > 1$, problem $(\mathcal{N}_\lambda)$ has at least two non-zero, non-negative $G-$invariant weak solutions.*

(b) *For every $G \in \mathcal{G}_{N,m}$, there exists $\lambda_G > 1$ such that $(\mathcal{N}_{\lambda_G})$ has at least three non-zero, non-negative $G-$invariant weak solutions.*

Remark 6.2 ***(a)*** *Since $f(0) = 0$, without loss of generality we can assume that f is defined on the whole real axis, putting $f(s) = 0$ for all $s \le 0$. In this case $F(s) = 0$ for all $s \le 0$, where F is defined in the Theorem 6.1* ***(i)****.*

(b) *Note that if $u \in W^{1,p}(\Omega)$ is a weak solution of $(\mathcal{N}_\lambda)$, then $u \ge 0$. Indeed, if we put $v = u_- = \min\{0, u\}$ as a test function in (6.3.1), then from*

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla u_- + |u|^{p-2}uu_-)dxdy = \lambda \int_\Omega \alpha(x,y)f(u)u_- dxdy = 0,$$

it follows that $u_- = 0$.

(c) *Note that beside of the above assumptions, if*

$$\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0,$$

then for small values of $\lambda > 0$, problem $(\mathcal{N}_\lambda)$ has only the trivial solution. Indeed, if $u \in W^{1,p}(\Omega)$ is a weak solution of $(\mathcal{N}_\lambda)$, and we put as the test function $v = u$ in relation (6.3.1), one obtains

$$\int_\Omega (|\nabla u|^p + |u|^p)dxdy = \lambda \int_\Omega \alpha(x,y)f(u)udxdy \le \lambda c_f \|\alpha\|_{L^\infty} \int_\Omega |u|^p dxdy,$$

where $c_f = \max\limits_{s>0} \dfrac{|f(s)|}{s^{p-1}} > 0$. Therefore, if $\lambda < (c_f\|\alpha\|_{L^\infty})^{-1}$, then $u = 0$.

A crucial step in our investigation is the following theorem:

Theorem 6.3 *Let $p > N \ge m + 2$, $\Omega = \omega \times \mathbb{R}^{N-m}$, and $G \in \mathcal{G}_{N,m}$. Then the embedding $W_G^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.*

6.1.2 Proof of Theorem 6.1

Throughout this subsection, we fix $G \in \mathcal{G}_{N,m}$ and let $\mathcal{L}^G = \mathcal{L}|_{W_G^{1,p}(\Omega)}$. We assume that all assumptions of Theorem 6.1 are satisfied. Since $\alpha \in L^\infty(\Omega)$ has a compact support, then $\alpha \in L^1(\Omega)$ as well.

Proposition 6.4

(a) *The functional $\mathcal{L}^G$ is weakly lower semicontinuous.*

(b) *For every $\lambda \geq 0$, $\mathcal{E}_\lambda^G$ satisfies the Palais-Smale condition.*

Corollary 6.5 *Let $p > N \geq m+2$, $\Omega = \omega \times \mathbb{R}^{N-m}$, $\alpha \in L^\infty(\Omega)$ be a cylindrically symmetric, non-negative, non-zero function with compact support K. Let $\mu = \operatorname{dist}(K, \partial\Omega)$, $f : [0, +\infty[\to \mathbb{R}$ be a continuous function such that $f(0) = 0$. Assume that*

j) $M_F = \sup F_{[0,+\infty[} < +\infty$;

jj) *zero is a local maximum of F;*

jjj) *there exists constants r, δ such that*

$$0 < r < \min\left\{\mu, \lim_{p\to+\infty}\left[\frac{|K_\mu \setminus K|}{\mu^p} + |K_\mu|\right]^{-\frac{1}{p}}\right\}$$

and $\delta > 2$ such that

$$0 < F(r) = \max_{[0,\delta]} F < M_F.$$

Then, there exists $p_0 > N$ such that for each $p \geq p_0$ both conclusions of Theorem 6.1 hold.

Example 6.6 *Let us choose $\Omega = [-1, 1] \times \mathbb{R}^2$ and*

$$K = \left\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq \frac{1}{4}\right\}.$$

Then, we can see that $\mu = dist(K, \partial\Omega) = \frac{1}{2}$ and $|K| = \frac{\pi}{6}$, $|K_\mu| = \frac{4}{3}\pi$.

We assume that

$$||\alpha||_{L^1} > \frac{75\pi^2}{2^7}. \tag{6.1.4}$$

Let us consider the function $g : [0, +\infty[\to \mathbb{R}, g(x) = \sin^5(2\pi x)$ and a positive integer $k_0 \in \mathbb{N}$, such that $k_0 \geq c_\infty \sqrt[4]{4\|\alpha\|_{L^1}}$ and a function $h : [0, +\infty[\to \mathbb{R}$

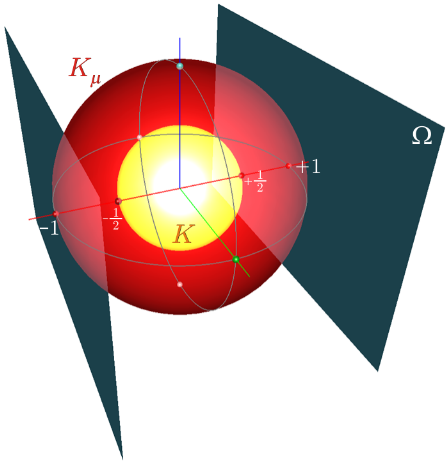

Figure 6.1: Ω, K and K_μ

such that $h(x) = \frac{3}{2} - 6\left(x - k_0 + \frac{1}{2}\right)^2$. *Now we are in the position to construct the following function:*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - \delta_{k,k_0}) \cdot \chi_{[k-1,k]}(x) \cdot g(x) + \delta_{k,k_0} \cdot \chi_{[k_0-1,k_0]}(x) \cdot h(x), \ (\forall) x \geq 0,$$

where $\chi_{[k-1,k]}$ *is the characteristic function of the interval* $[k-1, k]$*, and* δ_{k,k_0}

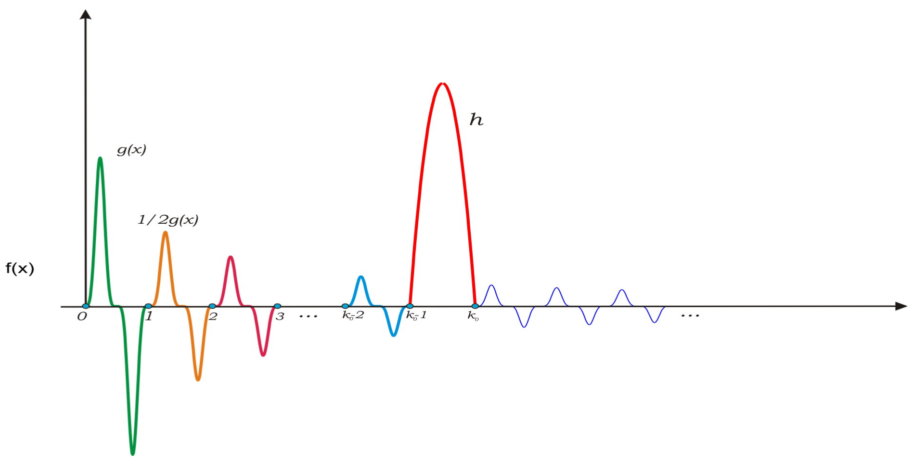

Figure 6.2: Graph of the function f

is the Kronecker's delta, i.e., $\delta_{k,k_0} = \begin{cases} 0, \text{ if } k \neq k_0 \\ 1, \text{ if } k = k_0 \end{cases}$*. More explicitly,*

$$f(x) = \begin{cases} \frac{1}{k}\sin^5(2\pi x), & \text{if } x \in [k-1,k], k \neq k_0 \\ \frac{3}{2} - 6\left(x - k_0 + \frac{1}{2}\right)^2, & \text{if } x \in [k_0-1,k_0]. \end{cases}$$

First of all, we show that $F(x) = \int_0^x f(t)dt$ *is bounded from above. Indeed, from the construction it yields that*

$$\max_{x \in \mathbb{R}_+ \setminus [k_0-1,k_0]} F(x) = \max_{x \in [0,1]} F(x) = \int_0^{\frac{1}{2}} \sin^5(2\pi x)dx = \frac{8}{15\pi}.$$

On the other hand,

$$\max_{x \in [k_0-1,k_0]} F(x) = \int_{k_0-1}^{k_0} \left[\frac{3}{2} - 6\left(x - k_0 + \frac{1}{2}\right)^2\right] dx = 1,$$

which means, that F *is bounded from above and* $M_F = 1$*.*

One has

$$\lim_{s \to 0^+} \frac{f(s)}{s^3} = \lim_{s \to 0^+} \frac{\sin^5(2\pi s)}{s^3} = 0. \tag{6.1.5}$$

By choosing $p = 4$*,* $N = 3$ *and taking into account (6.1.5) and* $M_F = 1$*, it is easy to see that* F *satisfies the assumptions **(i)** and **(ii)**. Inequalities (6.1.4) and* $k_0 \geq c_\infty \sqrt[4]{4\|\alpha\|_{L^1}}$ *guaranty that the assumptions **(iii)** and **(iv)** of Theorem 6.1 hold. In conclusion, we can apply Theorem 6.1.*

6.2 A multiplicity result on strip-like domain

In this section we would like to apply the same method which was presented above, for the following quasilinear equation coupled with a Neumann boundary condition

$$\begin{cases} -\Delta_p u + |u|^{p-2} \cdot u = \beta(x,y)g(u(x,y)) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = \lambda\alpha(x,y)f(u(x,y)) & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{N}_\lambda$}$$

here again $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being with smooth boundary, $p > N$, Δ_p is the p-Laplacian operator, λ is a positive parameter, $\alpha \in L^\infty(\partial\Omega) \cap L^1(\partial\Omega)$, $\beta \in L^1(\Omega)$ are non-zero positive potentials, and $f, g : [0, +\infty[\to \mathbb{R}$ are continuous functions with $f(0) = g(0) = 0$.

Problems of the type $(\mathcal{N}_\lambda)$ have been the object of intensive investigations in the recent years; see [20] and references therein. In a very recent paper, F. Faraci and A. Kristály (see [20]) proved a multiplicity result for a model quasilinear Dirichlet problem depending on a positive parameter. More precisely the authors studied the problem

$$\begin{cases} -\Delta_p u = \lambda\alpha(x)f(u) & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where Ω is a bounded open connected set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p > n$, λ is a positive parameter, $\alpha \in L^1(\Omega)$ is a non-zero potential. In the aforementioned paper, the authors guarantee the existence of at least two non-trivial weak solutions of $(\mathcal{P}_\lambda)$ in $W_0^{1,p}(\Omega)$ for $\lambda > 0$ large enough, and the existence of a parameter $\tilde{\lambda} > 0$ for which $(\mathcal{P}_{\tilde{\lambda}})$ has at least three non-zero weak solutions. They also discussed the sharpness of the last statement.

In the sequel, we outline our approach and state the main result. Without loss of generality we can assume that f, g are defined on the whole real axis, putting $f(s) = g(s) = 0$ for all $s \leq 0$. Let $F, G : \mathbb{R} \to \mathbb{R}$ be the primitives of f and g, i.e.,

$$F(s) = \int_0^s f(t)dt \text{ and } G(s) = \int_0^s g(t)dt.$$

Let q, r such that $q < p < r$ and we assume that the following hypotheses hold:

(A) $\limsup_{|s| \to 0} \frac{F(s)}{|s|^r} < \infty$ and $|f(s)| \leq C(1 + |s|^{q-1})$;

(B) There exists $\delta_1 > 0$ such that

$$g(t) \leq 0, \forall 0 \leq t < \delta_1,$$

$$f(t) \leq 0, \forall 0 \leq t < \delta_1;$$

(C) There exist q, r such that $q < p < r$ and

$$\lim_{t \to 0} \frac{G(t)}{|t|^q} = 0$$

and

$$\lim_{t \to \infty} \frac{G(t)}{|t|^r} = 0;$$

(D) Let $K_\tau = \{(x, y) \in \omega \times \mathbb{R}^{N-m} : ||y|| < \tau\}$. We assume that

$$\inf_{(x,y) \in K_\tau} \alpha(x, y) > 0;$$

(E) $p \cdot \tilde{c} \|\beta\|_1 c_\infty^p < 1$, where c_∞ is the embedding constant in $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, and $\widetilde{c} = \max_{t \in \mathbb{R}} \frac{G(t)}{|t|^p} < +\infty$ (see Lemma 6.12);

(F) There exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$.

Our main result of this section reads as follows:

Theorem 6.7 *Assume that $p > N \geq 2$, and let $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega \subset \mathbb{R}^m$ is a bounded open domain with smooth boundary. Let $\alpha \in L^\infty(\partial\Omega) \cap L^1(\partial\Omega)$, $\beta \in L^1(\Omega)$ be two positive cylindrically symmetric functions, and let $f, g : [0, +\infty[\to \mathbb{R}$ be continuous functions with $f(0) = g(0) = 0$, satisfying $(\mathbf{A}) - (\mathbf{F})$. Then, there exists a $\lambda_0 > 0$ such that, for every $\lambda > \lambda_0$ problem $(\mathcal{N}_\lambda)$ has at least two cylindrically symmetric non-zero, weak solutions.*

Remark 6.8 Under the assumptions of Theorem 6.7, one can choose

$$\lambda_0 = \inf\left\{ \frac{\frac{1}{p}||u||^p - J(u)}{\Phi(u)} : u \in W_c^{1,p}(\Omega), \Phi(u) > 0 \right\}.$$

Example 6.9 Let us choose

$$f(s) = \ln(1 + (s-1)_+^r),$$

and

$$g(s) = rs^{r-1}(1-s)_+.$$

Then the assumptions **(A),(B),(C),(F)** hold immediately, where $t_+ = \max\{0, t\}$.

In this case we have that

$$\widetilde{c} = \frac{1}{r+1-p} \cdot \left(\frac{(r-p)(r+1)}{r(r+1-p)}\right)^{r-p}.$$

Lemma 6.10 *For every $\lambda > 0$, the functional $\mathcal{E}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ is continuously differentiable.*

Lemma 6.11 *There exists $\varepsilon_f > 0$ such that $|F(s)| \leq \varepsilon_f |s|^r$ for every $s \in \mathbb{R}$.*

Lemma 6.12 *We have that*

$$\max_{t \in \mathbb{R}} \frac{G(t)}{|t|^p} < \infty.$$

Lemma 6.13 *Let $\lambda > 0$ be arbitrary fixed. Then every bounded sequence $\{u_n\} \subset W_c^{1,p}(\Omega)$ such that*

$$\|\mathcal{F}_\lambda(u_n)\|_{W_c^{1,p}(\Omega)^*} \to 0,$$

contains a strongly convergent subsequence.

Lemma 6.14 *For every $\lambda \geq 0$, the functional $\mathcal{F}_\lambda : W_c^{1,p}(\Omega) \to \mathbb{R}$ is coercive.*

Lemma 6.15 *The functional $\mathcal{F}_\lambda : W_c^{1,p}(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale condition.*

6.3 A sublinear differential inclusion on strip-like domains

6.3.1 Formulation of the problem

In this section we consider the following sublinear elliptic differential inclusion problem coupled with a zero Dirichlet boundary condition

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda\alpha(x,y)\partial F(u(x,y)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being bounded and open with smooth boundary, $p > N$, $N - m \geq 2$, λ is a positive parameter, $\alpha \in L^\infty(\Omega) \cap L^1(\Omega)$ is a axially symmetric, non-negative, non-zero function and ∂F stands for the generalized gradient of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. The aforementioned problem is interesting not only from mathematical point of view but for its applicability in mathematical physics as well (e.g., in the theory of fluid mechanics) where solutions of elliptic problems correspond to certain equilibrium state of the physical system. We refer the reader to the works [15]- [16], [23]- [39], [41], [49], [72]. The motivation to consider this kind of inclusion comes from discontinuous phenomena. Namely, if $f \in L^\infty_{\text{loc}}(\mathbb{R})$ is not necessarily continuous, the problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda\alpha(x,y)f(u(x,y)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}'_\lambda$}$$

needs not to have a solution, due to the presence of certain gaps in the right hand side. However, if we replace the function f in $(\mathcal{P}'_\lambda)$ by an interval $[\underline{f}(\,\cdot\,), \overline{f}(\,\cdot\,)]$, where

$$\underline{f}(s) = \lim_{\delta \to 0^+} \operatorname{essinf}_{|t-s|<\delta} f(t)$$

and

$$\overline{f}(s) = \lim_{\delta \to 0^+} \operatorname{esssup}_{|t-s|<\delta} f(t),$$

the new set-valued problem may have solutions in a certain sense. Moreover, if $F(s) = \int_0^s f(t)dt$ with $f \in L^\infty_{\text{loc}}(\mathbb{R})$, then F is locally Lipschitz and $\partial F(s) = [\underline{f}(s), \overline{f}(s)]$ for every $s \in \mathbb{R}$, see [40]. We study problem $(\mathcal{P}_\lambda)$ by using variational arguments which require certain compactness of embeddings. Usually, when we are dealing with bounded domains, Sobolev spaces can be compactly embedded into various Lebesgue spaces. However, when the domain is unbounded, no compact embedding can be expected of the Sobolev spaces due to dilations of translations. Consequently, in order to study our problem we need a compact embedding theorem which will exploit the symmetry of the strip-like domain, described recently in the paper [21] or [23] or Theorem 6.3.

As a result, we obtain some weak solutions with respect to the narrowed space $W^{1,p}_c(\Omega)$ (see (6.1.1)). Note that $W^{1,p}_c(\Omega)$ is a proper subspace of $W^{1,p}_0(\Omega)$, thus further arguments are needed to prove that the solutions are actually weak solutions of the problem with respect to the whole space $W^{1,p}_0(\Omega)$. The answer will be achieved by the so-called principle of symmetric criticality (see e.g. [55] for the smooth version). Recently, in [46], the authors extended this principle to perturbed locally Lipschitz functionals by a lower semicontinuous, proper and convex functional which will be useful in our investigations. As we already pointed out, the main objective of this section is to ensure the existence of solutions for problem $(\mathcal{P}_\lambda)$ where the natural functional space is the Sobolev space $W^{1,p}_0(\Omega)$. In order to present our main result, we first recall that $u \in W^{1,p}_0(\Omega)$ is a *weak solution* of problem $(\mathcal{P}_\lambda)$ if for all $v \in W^{1,p}_0(\Omega)$ there exists $\xi_F \in \partial F(u(x,y))$ for a.e. $(x,y) \in \Omega$ such that

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dxdy = \lambda \int_\Omega \alpha(x,y)\xi_F v(x,y)dxdy. \tag{6.3.1}$$

In the sequel, we outline our approach and state the main result. We assume that the following hypotheses hold:

(A) $\inf_{\omega \times B(0,R)} \alpha(x,y) > 0$, where
$B(0,R) = \{x \in \mathbb{R}^{N-m} : \|x\|_{\mathbb{R}^{N-m}} < R\} \subset \mathbb{R}^{N-m}$;

$(\mathcal{F}_1)$ $\lim_{|s|\to 0} \dfrac{\max\{|\xi| : \xi \in \partial F(s)\}}{|s|^{p-1}} = 0;$

$(\mathcal{F}_2)$ $\lim\limits_{|s|\to+\infty} \dfrac{F(s)}{|s|^{p-1}} = 0;$

$(\mathcal{F}_3)$ There exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$.

The main result of this section reads as follows:

Theorem 6.16 *Assume that $p > N \geq 2$, and let $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega \subset \mathbb{R}^m$ is a bounded open domain with smooth boundary. Let $\alpha \in L^\infty(\Omega) \cap L^1(\Omega)$ be a positive cylindrically (axially) symmetric function satisfying (A), and let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function with $F(0) = 0$, satisfying $(\mathcal{F}_1) - (\mathcal{F}_3)$. Then there exists a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ the problem $(\mathcal{P}_\lambda)$ has at least two axially symmetric non-zero, weak solutions in $W_0^{1,p}(\Omega)$.*

The proof of this theorem is based on variational arguments. To see this, we consider the functionals $I, \mathcal{L} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\|u\|^p, \mathcal{L}(u) = \int_\Omega \alpha(x,y)F(x,y)dxdy.$$

Here, $\|\cdot\|$ denotes the standard norm on $W_0^{1,p}(\Omega)$. The energy functional associated with problem $(\mathcal{P}_\lambda)$ is given by

$$\mathcal{E}_\lambda(u) = I(u) - \lambda\mathcal{L}(u),$$

which is a locally Lipschitz functional on $W_0^{1,p}(\Omega)$. Furthermore, a standard argument shows that the critical points (in the sense of Chang) of $\mathcal{E}_\lambda$ are precisely the weak solutions of the problem $(\mathcal{P}_\lambda)$. Moreover, due to the non-smooth principle of symmetric criticality of Palais (see [46]), the critical points of $\mathscr{A}_\lambda = \mathcal{E}_\lambda|_{W_c^{1,p}(\Omega)}$ become critical points of $\mathcal{E}_\lambda$ as well, so axially symmetric, weak solutions of the problem $(\mathcal{P}_\lambda)$. Therefore, it is enough to guarantee critical points for $\mathscr{A}_\lambda = \mathcal{E}_\lambda|_{W_c^{1,p}(\Omega)}$ where the compactness of the embedding $W_c^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ will be deeply exploited.

Remark 6.17 Note that if the condition $(\mathcal{F}_2)$ is replaced by

$(\mathcal{F}_2)'$ $\lim\limits_{|s|\to\infty} \dfrac{\max\{|\xi| : \xi \in \partial F(s)\}}{|s|^{p-1}} = 0,$

then for small values of $\lambda > 0$, problem $(\mathcal{P}_\lambda)$ has only the trivial solution. Indeed, if $u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathcal{P}_\lambda)$, and we put as the test function $v = u$ in relation (6.3.1), one obtains

$$\int_\Omega (|\nabla u|^p + |u|^p)dxdy = \lambda \int_\Omega \alpha(x,y)\xi_F u dxdy \leq$$

$$\leq \lambda c_F \|\alpha\|_{L^\infty} \int_\Omega |u|^p dxdy,$$

where $c_F = \max_{s>0} \dfrac{\max\{|\xi| : \xi \in \partial F(s)\}}{s^{p-1}} > 0$. Therefore, if $\lambda < (c_F\|\alpha\|_{L^\infty})^{-1}$, then $u = 0$.

6.3.2 Proof of Theorem 6.16

Proposition 6.18 *The functional $\mathscr{A}_\lambda : W_c^{1,p}(\Omega) \to \mathbb{R}$ is coercive for every $\lambda > 0$.*

Proposition 6.19 *For every $\lambda > 0$, $\mathscr{A}_\lambda$ satisfies the non-smooth Palais-Smale condition.*

7

Symmetrically invariant multiple solutions

In the present chapter we prove a multiplicity result for a quasi-linear elliptic system, coupled with a homogeneous Dirichlet boundary condition $(\mathcal{S}_\lambda)$ *on the unit ball, depending on a positive parameter* λ*. By variational methods, we prove that for large values of* λ*, the problem* $(\mathcal{S}_\lambda)$ *has at least two non-zero symmetric invariant weak solutions.*

Contents

7.1 Formulation of the problem

Through this chapter we consider the following quasi-linear, elliptic differential system coupled with a homogeneous Dirichlet boundary condition,

$$\begin{cases} -\Delta_p u = \lambda F_u(x, u(x), v(x)) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x, u(x), v(x)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{S}_\lambda$}$$

where λ is a positive parameter and $N > p, q > 1$, $\Omega = B(0,1) \subset \mathbb{R}^N$ is the unit ball, $F \in \mathcal{C}^1(\Omega \times \mathbb{R}^2, \mathbb{R})$, F_z denotes the partial derivative of F with respect to z. Systems of type $(\mathcal{S}_\lambda)$ have been the object of intensive investigations on bounded domains. We refer to the works of Boccardo and de Figueiredo [9], de Figueiredo [30], de Nápoli, Mariani [51] and Kristály, Rădulescu, Varga [43].

From the articles dealing with systems, we would like to highlight the paper of A. Kristály and I. Mezei, see [42], which studies a gradient-type system defined on a strip-like domain, depending on two parameters, and proving a Ricceri-type three critical point result. While keeping some conditions from [42], we also aim to give a multiplicity theorem for our problem.

As we alredy have pointed out, our aim is to examine the above problem in the point of view of symmetrizations, namely to prove a result which ensures the existence of symmetrically invariant solutions.

The aforementioned problem is interesting not only from a mathematical point of view but also from its applicability in mathematical physics. Problem $(\mathcal{S}_\lambda)$ is a generalization of the equation of the spring pendulum. A spring pendulum is a physical system where a piece of mass is connected to a spring so that the resulting motion contains elements of a simple pendulum motion as well as a spring motion. The equation of spring pendulum is the following:

$$\begin{cases} -\ddot{x}(t) = \omega_0^2 x(t) \left(1 - \frac{l_0}{\sqrt{x(t)^2 + y(t)^2}}\right) \\ -\ddot{y}(t) = \omega_0^2 y(t) \left(1 - \frac{l_0}{\sqrt{x(t)^2 + y(t)^2}}\right), \end{cases} \tag{$\mathcal{S}$}$$

where $\omega_0 = \sqrt{\frac{g}{l_0}}$ and l_0 is the length of the spring at rest. A simple simulation shows how our numerical solutions can be represented, and that the orbit of this kind of pendulum has a fractal-like shape. Such phenomena are often studied in chaos theory.

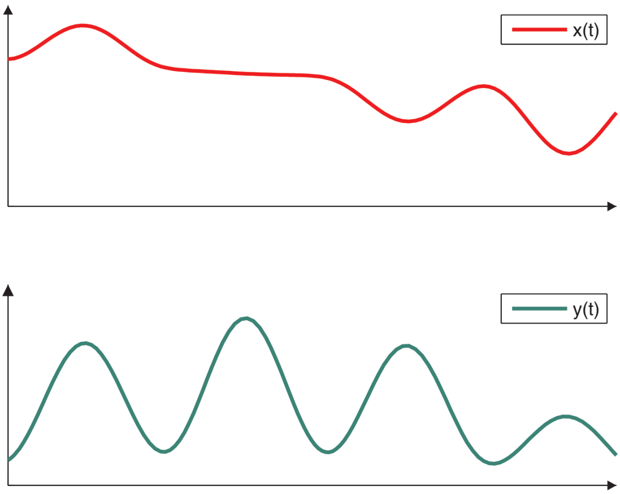

Figure 7.1: Solutions of spring pendulum

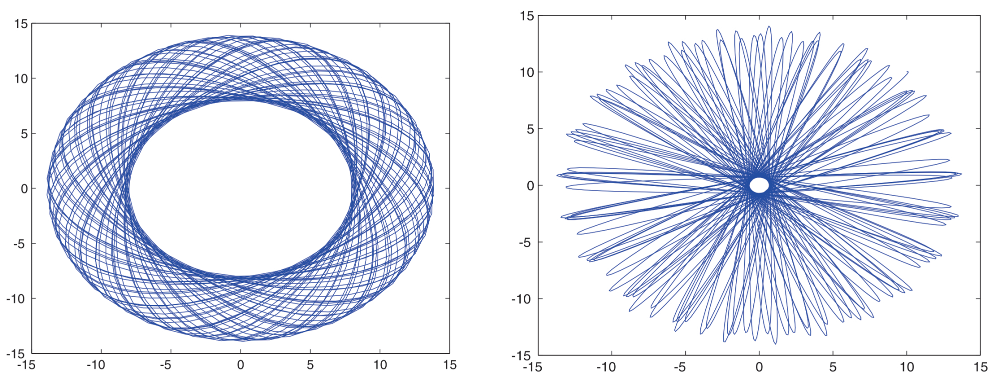

Figure 7.2: Orbit of the solutions

Problem $(\mathcal{S})$ can be treated as a variational problem, if we choose

$$F(x,y) = \frac{\omega_0^2(x^2+y^2)}{2} - \omega_0^2 \cdot l_0\sqrt{x^2+y^2},$$

then the energy functional associated with problem $(\mathcal{S})$ is defined by

$$E(x,y)=\int_I (x')^2+(y')^2dt-\int_I F(x,y)dt,$$

where $I\subset\mathbb{R}_+$.

The main objective of this chapter is to ensure the existence of symmetric invariant non-trivial solutions for problem $(\mathcal{S}_\lambda)$ where the natural functional framework is the Sobolev space $W_0^{1,p,q}(\Omega)=W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

In order to present our main result, we first recall that $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ is a *weak solution* of problem $(\mathcal{S}_\lambda)$ if

$$\begin{cases}\displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u\nabla w_1 dx-\lambda\int_\Omega F_u(x,u(x),v(x))w_1(x)dx=0\\[2ex] \displaystyle\int_\Omega |\nabla v|^{q-2}\nabla v\nabla w_2 dx-\lambda\int_\Omega F_v(x,u(x),v(x))w_2(x)dx=0,\end{cases}\tag{7.1.1}$$

for every $(w_1,w_2)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Through this chapter we consider the space $W_0^{1,\alpha}(\Omega)$ endowed with the norm

$$\|u\|_{1,\alpha}=\left(\int_\Omega |\nabla u|^\alpha\right)^{1/\alpha}\quad \alpha\in\{p,q\},$$

and for $\beta\in[\alpha,\alpha^*]$ we have the Sobolev embeddings $W_0^{1,\alpha}(\Omega)\hookrightarrow L^\beta(\Omega)$. The product space $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ is endowed with the norm $\|(u,v)\|_{1,p,q}=\|u\|_{1,p}+\|v\|_{1,q}$. We will denote by C_z the best Sobolev constant in the embedding $W_0^{1,z}(\Omega)\hookrightarrow L^z(\Omega)$. We assume that the following hypotheses hold:

$(\mathcal{F}_1)$ $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function, $(s,t)\mapsto F(x,s,t)$ is of $\mathcal{C}^1(\Omega\times\mathbb{R}^2,\mathbb{R})$ and $F(x,0,0)=F(x,s,0)=F(x,0,t)=0$ and $F_s(x,s,t)\cdot s_-+F_t(x,s,t)\cdot t_-\leq 0$ for all x,s,t, where $\tau_-=\min\{0,\tau\}$;

$(\mathcal{F}_2)$ $\displaystyle\lim_{(s,t)\to(0,0)}\frac{F(x,s,t)}{|s|^p+|t|^q}=0$, uniformly for every $x\in\Omega$;

$(\mathcal{F}_3)$ $\displaystyle\lim_{|s|+|t|\to+\infty}\frac{F(x,s,t)}{|s|^p+|t|^q}=0$, uniformly for every $x\in\Omega$;

($\mathcal{F}_4$) There exists, $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\int_\Omega F(x, u_0(x), v_0(x))dx > 0;$$

($\mathcal{F}_5$) For $F(x, s, t) = F(y, s, t)$ for each $x, y \in \Omega$ with $|x| = |y|$ and $s, t \in \mathbb{R}$ and for $x \in \Omega$ and $a \leq b$ and $c \leq d$

$$F(x, a, c) + F(x, b, d) \geq F(x, a, d) + F(x, b, c);$$

($\mathcal{F}_6$) For all x, s, t one has

$$F(x, s, t) \leq F(x, |s|, |t|).$$

Our main result reads as follows:

Theorem 7.1 *Assume that $p, q > 1$, and let $\Omega \subset \mathbb{R}^N$ be the unit ball. Let $F \in \mathcal{C}^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ be a function which satisfies $(\mathcal{F}_1) - (\mathcal{F}_6)$. There exists a λ_0 such that, for every $\lambda > \lambda_0$ problem $(\mathcal{S}_\lambda)$ has at least two weak solutions in $W_0^{1,p,q}(\Omega)$, invariant by spherical cap symmetrization.*

Remark 7.2 *Let $p = q = 2$, then the function $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $F(x, s, t) = \|x\| \ln(1 + s_+^2 \cdot t_+^2)$ fulfills the hypotheses $(\mathcal{F}_1) - (\mathcal{F}_6)$, where $\tau_+ = \max\{0, \tau\}$.*

Remark 7.3 *From $(\mathcal{F}_1)$ and $(\mathcal{F}_5)$ one can conclude the following inequality:*

$$F(x, 0, 0) + F(x, s, t) \geq F(x, 0, t) + F(x, s, 0), \text{ for } t, s \geq 0,$$

therefore

$$F(x, s, t) \geq 0 \text{ for } s, t \geq 0.$$

Remark 7.4 *If*

$$S_F = C_{\max} \sup_{(s,t)\neq(0,0)} \frac{|sF_s(x, s, t) + tF_t(x, s, t)|}{|s|^p + |t|^q} < \infty,$$

then there exists a λ_F *such that, for every* $0 < \lambda \leq \lambda_F$ *the problem* $(\mathcal{S}_\lambda)$ *has only the trivial solution. Indeed, a solution of* $(\mathcal{S}_\lambda)$ *is a pair* $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *such that*

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w_1 dx - \lambda \int_\Omega F_u(x,u(x),v(x))w_1(x))dx = 0 \\ \\ \displaystyle\int_\Omega |\nabla v|^{q-2}\nabla v \nabla w_2 dx - \lambda \int_\Omega F_v(x,u(x),v(x))w_2(x)dx = 0, \end{cases}$$

for all $w_1 \in W_0^{1,p}(\Omega)$ *and* $w_2 \in W_0^{1,q}(\Omega)$*.*

Choosing $w_1 = u$ *and* $w_2 = v$*, we obtain that*

$$\|u\|_{1,p}^p + \|v\|_{1,q}^q = \lambda \int_\Omega (F_u(x,u,v)u + F_v(x,u,v)v)dx \leq \lambda \frac{S_F}{C_{\max}} \int_\Omega (|u|^p + |v|^q) \leq$$

$$\leq \lambda \frac{S_F}{C_{\max}} (C_p^p \|u\|_{1,p}^p + C_q^q \|v\|_{1,q}^q) \leq \lambda S_F (\|u\|_{1,p}^p + \|v\|_{1,q}^q),$$

where, $C_{\max} = \max\{C_p^p, C_q^q\}$*, therefore if* $\lambda < \frac{1}{S_F}$ *then we necessarily have that* $(u,v) = (0,0)$*, which concludes the proof of this remark.*

Remark 7.5 *Note that if* (u,v) *is a weak solution*

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_- dx - \lambda \int_\Omega F_u(x,u(x),v(x))u_-(x)dx = 0 \\ \\ \displaystyle\int_\Omega |\nabla v|^{q-2}\nabla v \nabla v_- dx - \lambda \int_\Omega F_v(x,u(x),v(x))v_-(x)dx = 0, \end{cases}$$

follows that $u_- = v_- = 0$*.*

Then energy functional associated with the problem $(\mathcal{S}_\lambda)$ is defined by

$$\mathscr{A}_\lambda(u,v) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q - \lambda \int_\Omega F(x,u,v)dx. \tag{7.1.2}$$

7.2 Proof of Theorem 7.1

Before proving our main result, we prove that our functional $\mathscr{A}_\lambda$ is coercive and satisfies the Palais-Smale condition on $W_0^{1,p,q}(\Omega) = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Lemma 7.6 *The functional $\mathscr{A}_\lambda : W_0^{1,p,q}(\Omega) \to \mathbb{R}$ is coercive for every $\lambda \geq 0$.*

Lemma 7.7 *One has,*

$$\mathscr{A}_\lambda(u^H, v^H) \leq \mathscr{A}_\lambda(u, v).$$

Remark 7.8 *Using the Sobolev embeddings, $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ and $(\mathcal{F}_4)$, one can prove in a standard way that $\mathcal{F}$ is of class $\mathcal{C}^1$, its differential being*

$$\mathcal{F}(u,v)(w,y) = \int_\Omega [F_u(x,u,v)w + F_v(x,u,v)y]dx,$$

for every $u, w \in W_0^{1,p}(\Omega)$ and $v, y \in W_0^{1,q}(\Omega)$.

Lemma 7.9 *Let $\lambda \geq 0$ be fixed and let $\{(u_n, v_n)\}$ be a bounded sequence in $W_0^{1,p,q}(\Omega)$ such that*

$$\|\mathscr{A}_\lambda'(u_n, v_n)\|_\star \to 0$$

as $n \to \infty$. Then $\{(u_n, v_n)\}$ contains a strongly convergent subsequence in $W_0^{1,p,q}(\Omega)$.

8

Poisson-type equations on Finsler-Hadamard manifolds

By using direct methods from the calculus of variations, we establish uniqueness, location and rigidity results for the (singular) Poisson equation involving the Finsler-Laplace operator on Finsler-Hadamard manifolds having finite reversibility constant.

Contents

8.1 Formulation of the problem

Elliptic problems on Riemannian manifolds have been intensively studied in the last decades. On one hand, deep achievements have been done in connection with the famous Yamabe problem on Riemannian manifolds which can be transformed into an elliptic PDE involving the natural Laplace-Beltrami operator, see Aubin [3] and Hebey [36]. On the other hand, various anisotropic elliptic problems are discussed on Minkowski spaces of the form $(\mathbb{R}^n, F)$ where $F \in C^2(\mathbb{R}^n, [0, \infty))$ is convex and the leading term is given by the non-linear Finsler-Laplace operator associated with the Minkowski norm F, see Alvino, Ferone, Lions and Trombetti [1], Bellettini and Paolini [6], Belloni, Ferone and Kawohl [7], [29], and references therein.

Although in the aforementioned works the involved metrics are symmetric, *asymmetry* is abundant in real life. In order to describe such phenomena, we put ourselves into the context of not necessarily reversible Finsler manifolds.

The main objective of this chapter is twofold: (a) to describe some new, unexpected aspects of Sobolev spaces defined on non-compact Finsler manifolds; (b) to apply elements from the calculus of variations in the study of a singular Poisson equation involving the highly nonlinear Finsler-Laplace operator. In the sequel, we roughly present the main results of this chapter.

Let (M, F) be a Finsler manifold. In a standard manner, one can introduce the Sobolev spaces $W^{1,2}(M)$ and $W_0^{1,2}(M)$ associated with (M, F), see Ge and Shen [33], and Ohta and Sturm [54]. To be more precise, let

$$W^{1,2}(M) = \left\{ u \in W^{1,2}_{\mathrm{loc}}(M) : \int_M F^{*2}(x, Du(x)) \mathrm{d}V_F(x) < +\infty \right\},$$

and $W_0^{1,2}(M)$ be the closure of $C_0^\infty(M)$ in $W^{1,2}(M)$ with respect to the (positively homogeneous) norm

$$\|u\|_F = \left(\int_M F^{*2}(x, Du(x)) \mathrm{d}V_F(x) + \int_M u^2(x) \mathrm{d}V_F(x) \right)^{1/2}, \qquad (8.1.1)$$

where F^* denotes the polar transform of F. Although it is possible to use an arbitrarily measure on (M, F) to define Sobolev spaces (see [54]), here and in the sequel, we shall use the canonical Hausdorff measure $\mathrm{d}V_F(x)$ on (M,F). When M is *compact*, we know from [33] and [54] that the Sobolev space $W_0^{1,2}(M)$ is a reflexive, complete normed vector space with a suitable norm. However, when M is *non-compact*, pathological situations may occur; in spite of the fact that $W^{1,2}(M)$ and $W_0^{1,2}(M)$ are closed convex cones, we shall show

Theorem 8.1 *Neither $W^{1,2}(M)$ nor $W_0^{1,2}(M)$ has necessarily a vector space structure.*

In fact, we prove that in general

$$u \in W_0^{1,2}(M) \not\Leftrightarrow -u \in W_0^{1,2}(M).$$

The latter issue will be explained on the two-dimensional Finsler-Poincaré disc model which is a non-compact, forward (but not backward) complete Randers space having its reversibility constant $r_F = +\infty$. Here,

$$r_F = \sup_{x\in M} \sup_{y\in T_xM\setminus\{0\}} \frac{F(x,y)}{F(x,-y)}.$$

It turns out however, that for Finsler manifolds with reversibility constant $r_F < +\infty$, $W_0^{1,2}(M)$ is a reflexive Banach space endowed with a suitable norm, equivalent to $\|\cdot\|_F$ from (8.1.1), see Theorem 8.4.

In the second part we consider that (M,F) is an $n-$dimensional Finsler-Hadamard manifold (i.e., simply connected, complete with non-positive flag curvature), $n \geq 3$, having its uniformity constant $l_F > 0$ (which implies in particular that $r_F < +\infty$). We shall study the model singular Poisson equation

$$\begin{cases} \boldsymbol{\Delta}(-u) - \mu \frac{u}{d_F^2(x_0,x)} = 1 & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}_\Omega^\mu$}$$

where $\boldsymbol{\Delta}$ denotes the Finsler-Laplace operator on (M,F), d_F is the metric function, $x_0 \in \Omega$ is fixed, $\mu \geq 0$ is a parameter, and $\Omega \subset M$ is an open and bounded domain with sufficiently smooth boundary. We prove that the singular energy functional associated with problem $(\mathcal{P}_\Omega^\mu)$ is strictly convex on $W_0^{1,2}(\Omega)$ whenever $\mu \in [0, l_F r_F^{-2}\overline{\mu})$, see Theorem 8.8; here, $\overline{\mu} = \frac{(n-2)^2}{4}$ is the optimal Hardy constant. By exploiting a comparison principle for the Finsler-Laplace operator and well known arguments from the calculus of variations, we prove that problem $(\mathcal{P}^\mu_\Omega)$ has a unique, non-negative weak solution whenever $\mu \in [0, l_F r_F^{-2}\overline{\mu})$, see Theorem 8.10.

Let $B^+(x_0,\rho) = \{x \in M : d_F(x_0,x) < \rho\}$ be the forward open geodesic ball with center x_0 and radius $\rho > 0$, and the profile function $\sigma_{\mu,\rho} : (0,\rho] \to \mathbb{R}$ given by

$$\sigma_{\mu,\rho}(s) = \frac{1}{\mu + 2n}\left(\rho^2\left(\frac{s}{\rho}\right)^{-\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}} - s^2\right). \tag{8.1.2}$$

Note that for every $\mu \in [0,\overline{\mu})$ and $\rho > 0$, the function $u^e_{\mu,\rho}(x) = \sigma_{\mu,\rho}(|x|)$ is the unique solution for $(\mathcal{P}^\mu_{B^e(0,\rho)})$ whenever $(M,F) = (\mathbb{R}^n, e)$ is the usual Euclidean space, where $|x|$ is the Euclidean norm and $B^e(0,\rho)$ is the Euclidean ball with center at the origin and radius $\rho > 0$. By combining anisotropic symmetrization arguments and a Bishop-Gromov-type comparison principle on Finsler-Hadamard manifolds, we can prove the following rigidity result:

Theorem 8.2 *Let (M,F) be an $n-$dimensional $(n \geq 3)$ Finsler-Hadamard manifold of Berwald type with $l_F > 0$. Let $x_0 \in M$ be fixed. Then the following statements are equivalent:*

(a) *For some $\mu \in [0, l_F r_F^{-2}\overline{\mu})$, the function $u_{\mu,\rho}(x) = \sigma_{\mu,\rho}(d_F(x_0,x))$ is the unique solution of the Poisson equation $(\mathcal{P}^\mu_{B^+(x_0,\rho)})$ for every $\rho > 0$;*

(b) *(M,F) is isometric to an $n-$dimensional Minkowski space.*

We shall establish a similar rigidity result also in the case of $3-$dimensional hyperbolic spaces for the non-singular Poisson equation, see Theorem 8.14.

Proposition 8.3 *Let (M,F) be a Finsler manifold. Then the following statements hold:*

(a) *If $l_F > 0$ then $r_F < +\infty$;*

(b) *If $r_F < +\infty$, then the forward and backward completeness of (M,F) coincide;*

(c) *If (M,F) is of Randers type with $\mathbf{S} = 0$ then $l_F > 0$.*

8.2 Reversibility versus Sobolev spaces

Theorem 8.4 *Let (M, F) be a complete, $n-$dimensional Finsler manifold. If $r_F < +\infty$ then $(W_0^{1,2}(M), \|\cdot\|_{F_s})$ is a reflexive Banach space.*

Remark 8.5 The statement of Theorem 8.4 remains valid for an arbitrary open domain $\Omega \subset M$ instead of the whole manifold M.

8.3 Convexity of the singular energy functional

In order to deal with singular problems of type $(\mathcal{P}_\Omega^\mu)$ we first need a Hardy inequality on (not necessarily reversible) Finsler-Hadamard manifold with $\mathbf{S} = 0$. As mentioned before, these spaces include Finsler-Hadamard manifolds of Berwald type (thus, both Minkowski spaces and Hadamard-Riemannian manifolds).

Proposition 8.6 *Let (M, F) be an $n-$dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S} = 0$, and let $x_0 \in M$ be fixed. Then*

$$\int_M F^{*2}(x, -D(|u|)(x))\mathrm{d}V_F(x) \geq \overline{\mu} \int_M \frac{u^2(x)}{d_F(x_0, x)^2}\mathrm{d}V_F(x),\ \forall u \in C_0^\infty(M), \tag{8.3.1}$$

where the constant $\overline{\mu} = \frac{(n-2)^2}{4}$ is optimal and never achieved.

Remark 8.7 Proposition 8.6 can be proved for an arbitrary open domain $\Omega \subset M$ instead of the whole manifold M with $x_0 \in \Omega$.

In the sequel, we prove the main result of this section.

Theorem 8.8 *Let (M,F) be an $n-$dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S}=0$ and $l_F>0$. Let $\Omega \subseteq M$ be an open domain and $x_0 \in \Omega$. Then the functional $\mathscr{K}_\mu : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by*

$$\mathscr{K}_\mu(u) = \int_\Omega F^{*2}(x, Du(x))\mathrm{d}V_F(x) - \mu \int_\Omega \frac{u^2(x)}{d_F(x_0,x)^2}\mathrm{d}V_F(x)$$

is positive and strictly convex whenever $0 \leq \mu < l_F r_F^{-2}\overline{\mu}$.

8.4 Singular Poisson equations

Let (M,F) be a (not necessarily reversible) complete, $n-$dimensional $(n \geq 3)$ Finsler manifold, and $\Omega \subset M$ be an open, forward bounded domain, $x_0 \in \Omega$. For $\mu \in \mathbb{R}$, on $W_0^{1,2}(\Omega)$ we define the *singular Finsler-Laplace operator*

$$\mathcal{L}_F^\mu u = \mathbf{\Delta}(-u) - \mu \frac{u}{d_F^2(x_0,x)}.$$

Note that in general $\mathbf{\Delta}(-u) \neq -\mathbf{\Delta}u$, unless $r_F = 1$.

Proposition 8.9 (Comparison principle) *Let (M,F) be an $n-$dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S}=0$ and $l_F>0$. Let $\Omega \subset M$ be an open domain. If $\mathcal{L}_F^\mu u \leq \mathcal{L}_F^\mu v$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ a.e. in Ω, whenever $\mu \in [0, l_F r_F^{-2}\overline{\mu})$.*

Let $\mu \in [0, l_F r_F^{-2}\overline{\mu})$ and $\kappa \in L^\infty(\Omega)$. We consider the singular Poisson problem

$$\begin{cases} \mathcal{L}_F^\mu u = \kappa(x) & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}_\Omega^{\mu,\kappa}$}$$

where $\Omega \subset M$ is an open, bounded domain. We introduce the *singular energy functional* associated with the operator $\mathcal{L}_F^\mu$ on $W_0^{1,2}(\Omega)$, defined by

$$\mathcal{E}_\mu(u) = (\mathcal{L}_F^\mu u)(u).$$

We have in fact

$$\mathcal{E}_\mu(u) = \int_\Omega F^{*2}(x, -Du(x))\mathrm{d}V_F(x) - \mu \int_M \frac{u^2(x)}{d_F(x_0, x)^2}\mathrm{d}V_F(x) = \mathscr{K}_\mu(-u).$$

Theorem 8.10 *Let (M, F) be an $n-$dimensional $(n \geq 3)$ Finsler-Hadamard manifold with $\mathbf{S} = 0$ and $l_F > 0$. Let $\Omega \subset M$ be an open, bounded domain and a non-negative function $\kappa \in L^\infty(\Omega)$. Then problem $(\mathcal{P}^{\mu,\kappa}_\Omega)$ has a unique, non-negative weak solution for every $\mu \in [0, l_F r_F^{-2}\overline{\mu})$.*

In the sequel, we focus our attention to Theorem 8.2. First, we have

Proposition 8.11 *Let $(M, F) = (\mathbb{R}^n, \|\cdot\|)$ be a Minkowski space and let $x_0 \in \mathbb{R}^n$ and $\mu \in [0, l_F r_F^{-2}\overline{\mu})$ be fixed. For every $\rho > 0$, the function $u_{\mu,\rho}(x) = \sigma_{\mu,\rho}(\|x - x_0\|)$ is the unique solution to problem $(\mathcal{P}^\mu_{B^+(x_0,\rho)})$.* (Recall that $\sigma_{\mu,\rho}$ is the profile function from (8.1.2).)

Remark 8.12 (i) In addition to the above facts, it is easy to see that

(a) $u_{\mu,\rho} \in C^1(B^+(x_0, \rho))$ if and only if $\mu = 0$, and

(b) $u_{\mu,\rho} \in C^2(B^+(x_0, \rho))$ if and only if $\mu = 0$ and $F = \|\cdot\|$ is Euclidean.

(ii) When $(M, F) = (\mathbb{R}^n, \|\cdot\|)$ is a reversible Minkowski space and $\mu = 0$, Proposition 8.11 reduces to Theorem 2.1 from Ferone and Kawohl [29].

We now state an estimate for the solution of the singular Poisson equation on *backward* geodesic balls on Minkowski spaces. Although problem $(\mathcal{P}^\mu_{B^-(x_0,\rho)})$ cannot be solved explicitly in general, the following sharp estimates can be given for its unique solution by means of the reversibility constant r_F and the profile function $\sigma_{\mu,\rho}$ from (8.1.2):

Proposition 8.13 *Let $(M, F) = (\mathbb{R}^n, \|\cdot\|)$ be a Minkowski space, let $x_0 \in \mathbb{R}^n$, $\mu \in [0, l_F r_F^{-2}\overline{\mu})$ and $\rho > 0$ be fixed. If $\tilde{u}_{\mu,\rho}$ denotes the unique solution to problem $(\mathcal{P}^\mu_{B^-(x_0,\rho)})$, then*

$$\sigma_{\mu, r_F^{-1}\rho}(\|x - x_0\|) \leq \tilde{u}_{\mu,\rho}(x) \leq \sigma_{\mu, r_F\rho}(\|x - x_0\|),\ x \in B^-(x_0, \rho).$$

Moreover, the above two bounds coincide if and only if (M, F) is reversible.

We conclude the chapter by establishing a rigidity result on hyperbolic spaces, similar to Theorem 8.2. According to Theorem 8.10, the Poisson equation $(\mathcal{P}^{\mu}_{B(x_0,\rho)})$ has a unique solution for every $\mu \in [0,\overline{\mu})$ in the case when $(M,F) = (\mathbb{H}^n, g_{\text{hyp}})$ is the well known $n-$dimensional hyperbolic space, $n \geq 3$.

Taking into consideration the above facts, we restrict our attention to the non-singular Poisson equation in $3-$dimensional Hadamard manifolds (i.e., simply connected, complete Riemannian manifolds with non-positive sectional curvature). More precisely, we consider the Poisson problem

$$\begin{cases} -\Delta_g u = 1 & \text{in} \quad B(x_0,\rho); \\ u = 0 & \text{on} \quad \partial B(x_0,\rho), \end{cases} \tag{$\mathcal{P}^0_{B(x_0,\rho)}$}$$

where $B(x_0,\rho)$ is the open geodesic ball with center $x_0 \in M$ and radius $\rho > 0$ in the $3-$dimensional Hadamard manifold (M,g).

For the $3-$dimensional hyperbolic space we use the Poincaré ball model $\mathbb{H}^3 = \{x \in \mathbb{R}^3 : |x| < 1\}$ endowed with the Riemannian metric $g_{\text{hyp}}(x) = (g_{ij}(x))_{i,j=1,\dots,n} = p(x)^2\delta_{ij}$, where $p(x) = \frac{2}{1-|x|^2}$. It is well known that $(\mathbb{H}^3, g_{\text{hyp}})$ is a Hadamard manifold with constant sectional curvature -1. Its canonical volume form is $\mathrm{d}V_{\mathbb{H}^3}(x) = p(x)^3\mathrm{d}x$, and in particular, the volume of the $3-$dimensional hyperbolic ball of radius $\rho > 0$ is

$$\mathrm{Vol}_{\mathbb{H}^3}(\rho) = \pi[\sinh(2\rho) - 2\rho].$$

The hyperbolic gradient and Laplace-Beltrami operator are given by

$$\nabla_{\mathbb{H}^3} u = \frac{\nabla u}{p^2} \text{ and } \Delta_{\mathbb{H}^3} u = p^{-3}\mathrm{div}(p\nabla u), \tag{8.4.1}$$

where ∇ denotes the Euclidean gradient in $\mathbb{R}^3$. The hyperbolic distance between the origin and $x \in \mathbb{H}^3$ is given by

$$d_{\mathbb{H}^3}(0,x) = \ln\left(\frac{1+|x|}{1-|x|}\right).$$

For $\rho > 0$, let $\nu_\rho : [0,\rho] \to \mathbb{R}$ be the profile function defined by

$$\nu_\rho(s) = \frac{\rho\coth(\rho) - s\coth(s)}{2}.$$

Theorem 8.14 *Let (M, g) be a $3-$dimensional Hadamard manifold and let $x_0 \in M$ be fixed. Then the following statements are equivalent:*

(a) *The function $u_\rho(x) = \nu_\rho(d_g(x_0, x))$ is the unique solution of the Poisson equation $(\mathcal{P}^0_{B(x_0,\rho)})$ for every $\rho > 0$;*

(b) *(M, g) is isometric to the $3-$dimensional hyperbolic space $(\mathbb{H}^3, g_{\text{hyp}})$.*

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