What is Mathematics ?

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Colleagues, in our profession it is meaningless to wish someone to be wealthy; however, to be healthy is most appropriate, particularly, after sixty when every additional year is a bonus from God. To a scientist even a small recognition works as a catalyst for him to continue in his research at least for some more time. Thus, I am really grateful to the administration of the University of Babes–Bolyai, Cluj–Napoca, and specially Professors Adrian Petrusel and Radu Precup for awarding me the title Professor Honoris Causa.

In the year 2014 my joint monograph number 35, entitled *Creators of Mathematical and Computational Sciences*, Springer, New York, pp. 495, appeared. This monograph, which is a systematic organization of the literature which I started in 1969,

• records the essential discoveries of mathematics in chronological order, following the birth of ideas on the basis of prior ideas ad infinitum,

• it examines contemporary events occurring side by side in different countries or cultures, reflecting some of the noblest thoughts of generations,

• it documents the winding path of mathematical scholarship throughout history, and most importantly, the thought process of each individual that resulted in the mastery of their subject,

• it implicitly addresses the nature and character of every mathematician as we try to understand their visible actions, and

• it offers amusing anecdotes and after dinner jokes which reveal the human nature of mathematicians, who are very often believed to be eccentric individuals.

The premise of our monograph is the following phenomenal quotations:

Eric Temple Bell (1883–1960) remarked, “It would be an injustice to pioneers in mathematics to stress modern mathematical ideas with little reference to those who initiated the first and possibly the most difficult steps. Nearly everything useful that was discovered in mathematics before the seventeenth century has either been so greatly simplified that it is now part of every regular school course, or it has long since been absorbed as a detail in some work of greater generality”.

Devotees of mathematics scrutinize, memorize, and derive formulas and theorems every day of their lives, but not many of them realize that the current level of mathematical knowledge has resulted from the strenuous labors of countless generations. One cannot underestimate the influence of every culture, personality, philosophy, region, religion, society, and social status on mathematical development throughout the centuries.

Hermann Hankel (1839–1873) observed, “In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new story to the old structure”.

Anthropologist Ralph Linton (1893–1953) stated hypothetically that “... if Albert Einstein (1879–1955) had been born into a primitive tribe which was unable to count beyond three, life–long application to mathematics probably would not have carried him beyond the development of a decimal system based on fingers and toes”.

Isaac Asimov (1920–1992) enunciated “Mathematics is a unique aspect of human thought, and its history differs in essence from all other histories. Only in mathematics there is no significant correction–only extensions. Each great mathematician adds to what came previously, but nothing needs to be uprooted”.

James Whitbread Lee Glaisher (FRS, FRAS) (1848–1928) commented “No subject loses more than mathematics by any attempt to dissociate it from its history”.

In the first 36 pages of the book we address the following fundamental questions:

- What is Mathematics?
- What is Mathematical Science?
- What is a Mathematical Proof?
- What is Computational Science/Mathematics?
- Is a proof in Computational Mathematics always possible?

The word 'mathematics' was coined by Pythagoras, who flourished around 500 BC. It meant 'a subject of instruction', and its first part, 'math', comes from an old Indo–European root that is related to the English word 'mind'. The Pythagoreans grouped arithmetic, astronomy, geometry, and music together and for several centuries mathematics referred to only these four subjects.

However, Pythagoras was not the creator of arithmetic, astronomy, and geometry, these subjects were extensively studied long before Pythagoras. Throughout the book we observe that a fair majority of the biggest breakthroughs in mathematics were made possible through the work of people other than those who have been credited in the history books. We recall the acute observation of

Sir Francis Darwin (1848–1925), "But in science the credit goes to the man who convinces the world, not to the man to whom the idea first occurs".

In many instances, the fault lies with the historians themselves, who inject their own opinion into their texts rather than reporting unbiased fact.

The origin of mathematics is the number sense, which has played a vital role in the development of these subjects.

Number sense can be defined as an intuitive understanding of the integers, their magnitude, their patterns and relationships, and how they are affected by the basic operations: addition, subtraction, multiplication, and division. The number sense, the ability to distinguish 'plenty' and 'few' without counting, is a useful tool for a conscious being and a fundamental ability of humans. For primitive man and children, mathematics is simply a comparison of small collections. So far as non–human living beings are concerned, there are recorded incidences of birds, animals, insects, and aquatic creatures who show through their behavior a certain number sense, namely, comparing/sorting. These creatures possibly possess mathematical abilities that exceed our observations or beliefs about them.

In antiquity, counting was considered a talent as mystical and arcane as casting spells and calling the Gods by name. In the Egyptian *Book of the Dead*, when a dead soul is challenged by Aqen, the ferryman who conveys departed spirits across a river in the netherworld, Aqen refuses to allow anyone aboard "who does not know the number of his fingers". The soul must then recite a counting rhyme to tally his fingers, satisfying the ferryman.

In the world, although there are about four thousand languages of which several hundred are widespread, there are only several dozen alphabets and writing systems that represent them; however, we can safely say that there is but one single place–value system that uses zero and nine numerals, symbolically $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, by which any number can be expressed and understood rather easily. This gift of India has truly united the universe in the language of numbers. It is worth noting that there is hardly any aspect of our life in which numbers do not play a significant–though generally hidden–part.

Arithmetic is the most useful and the simplest branch of all the sciences, and there is no other branch of human knowledge that has been more widely spread among the masses. It can be defined as the concrete knowledge of integers, rational numbers, irrational (illogical or unreasonable) numbers, real numbers, and mastery of their manipulation through the abstract operations. Thus arithmetic forms a very sophisticated realization of number sense. The ancient people of the Indus River Valley (which was home to more than five million people) had a highly developed knowledge of arithmetic and a sophisticated system of weights and measures.

Higher arithmetic, which is a synonym for **number theory**, is the branch of arithmetic that is concerned with the properties and relationships of whole numbers for their own sake, and hence has been classified as pure mathematics. Although the term 'number theory' (la theorie des nombres) was coined by Pierre de Fermat (1601–1665) during the seventeenth century, its history is at least 5000 years old. However, Pythagoras and his disciples are generally given credit for initiating the actual theory of numbers. Pythagoras believed that all universal laws can be explained by numbers, and hence he declared that "Everything is Number". The first systematic treatment of number theory was given by Euclid of Alexandria (around 325–265 BC) in books 7,8, and 9 of his Elements. The ancient Greeks were fascinated by number theory, in fact, Plato of Athens (around 427–347 BC) wrote that "We must endeavor to persuade those who are to be the principal men of the state to go and learn arithmetic".

Since antiquity, number theory has captivated the best minds of every era. During 1600–1920, Pafnuty Lvovich Chebyshev (1821–1894), Peter Gustav Lejeune Dirichlet (1805–1859), Leonhard Euler (1707–1783), Fermat, Karl Friedrich Gauss (1777–1855), Joseph Louis Lagrange (1736–1813), Adrien–Marie Legendre (1752–1833), and Srinivasa Ramanujan (1887–1920) greatly enriched number theory. In 1801, Gauss asserted that number is purely a psychic reality, a free creation of mind, and he crowned number theory as "The Queen of Mathematics". According to Bell, "number theory is the last great uncivilized continent of mathematics".

In recent years, number theory has been divided into: (1) elementary or classical number theory, (2) algebraic number theory, (3) analytical number theory, (4) probabilistic number theory, and (5) computational number theory, which finds diverse applications. An important aspect of number theory is that challenging problems can be formulated in very simple terms; however, hidden within their simplicity is complexity. Some of these problems have been instrumental in the development of large parts of mathematics. Amateurs and professionals are on an almost equal footing in this field.

Astronomy is one of the most ancient (about 3.4 million years before 4500 BC) continuously pursued branches of natural science. It can be broadly defined as the study of all objects beyond our world, especially celestial bodies such as Comets, Galaxies, Moons, Nebulae, Planets, Stars, and the Sun. Astronomy also helps us to understand the physical laws and origins of our universe. Early astronomy was limited to the observations, patterns, and predictions of the motions of objects visible to the naked eye. While the movements of the Planets have been tracked around the world, astronomy has deep roots in China, Egypt, India, Mesopotamia, Central America, and later in Greece.

The primitive man used number sense (and eventually arithmetic) to prove that celestial bodies exhibit regularity of behavior and move with identifiable patterns. He observed that the Moon passes from one phase to another (from new Moon to half Moon, from half Moon to full Moon) in 7 days, that the Moon goes around the Earth every 28 days, and that the Sun completes its orbit about the Earth in 365 days. These observations helped him make primitive calendars so that he could keep track of the beginning and end of the rainy monsoon season, the annual snow melt and the consequent rise of the rivers, auspicious days for performing sacrifices, and the pattern of solar eclipses.

In India more than 5000 years ago it was known that the distances of the Moon and Sun from the Earth are 108 (which is a sacred number in several eastern religions and cults) times the diameters of these heavenly bodies, respectively, (the observed values are 110.6 for the Moon and 107.6 for the Sun).

Astrology, which has been wrongly claimed to be older than astronomy, deals with the belief that there is a relationship between astronomical phenomena (especially the cycles of planets and the stars) and events in human life. Right from the beginning many cultures attached importance to astronomical events, particularly the Indians, the Chinese, and the Mayans, who all developed sophisticated systems for predicting terrestrial events based on celestial observations. Astrology was originally studied along with astronomy, but the two have been separated.

The word **Geometry** comes from two Greek words meaning 'earth' and 'measure', which was known 5000 years ago as *sulba/sulva*, or *rule of chords*. Considering an object, geometry examines its shape, measures its size, especially its length, area, and volume, and makes sense of its spatial relationship with other objects and the properties of its surrounding space. Geometry is the basis of the survey of the Earth and the heavens, of all astronomy, of most laws in physics, art, architecture, and painting, and of all engineering and technology; its utility is unquestioned.

The study of geometry originated in India for the purpose of constructing precisely measured altars for ritual sacrifices. Geometry was systematically studied by Egyptian priests because the periodical inundations of the Nile River obliterated property lines. Most of the historians of mathematics have falsely documented that "before 600 BC geometry lacked deductive structure, there were no theoretical results, nor any general rules of procedure. It supplied only calculations, and these sometimes approximate, for problems that had a practical bearing in construction and surveying".

Almost 5000 year old Sulvasutras list geometric constructions for squares, rectangles, parallelograms, and trapezia. It also shows how one may construct: a square n times in area to a given square; a square of area equal to the sum of two squares; a square whose whole area is equal to the difference of two squares; a square equal to a rectangle; a triangle equal to a rectangle; a triangle equal to a rhombus; and a square equal to the sum of two triangles or two pentagons. Several theorems are also explicitly mentioned: the diagonal of a rectangle divides it into equal parts; the diagonals of a rectangle bisecting each other and opposite areas are equal; the perpendicular through the vertex of an isosceles triangle on the base divides the triangle into equal halves; a rectangle and a parallelogram on the same base and between the parallels are equal in area; and the diagonals of a rhombus bisect each other at right angles.

Baudhayana contains one of the earliest references to what is known today as the Pythagorean theorem (with a convincing, valid proof). The Pythagorean triples $(3,4,5)$, $(5,12,13)$, $(8,15,17)$, and $(12,35,37)$ are part of Apastamba's rules for altar construction. Certain kinds of Pythagorean triples are listed on Plimpton 322 (a Babylonian clay tablet), which is believed to be almost 4000 years old. The Katyayana, written later, gives a general version of the Pythagorean theorem. The Pythagorean theorem was also known to the Chinese long before Pythagoras as the Gou–gu theorem.

Since the sixteenth century several different branches of geometry have emerged: affine geometry, algebraic geometry, analytic geometry/coordinate geometry, combinatorial geometry, computational geometry, descriptive geometry, differential geometry, discrete geometry, metric geometry, projective geometry, non-Euclidean geometry, and Riemannian geometry. Research in these geometries continues and from time to time revolutionary results are announced.

The word **Algebra** is derived from a Latin translation of the treatise *Hisab al–jabr w'al–muqabala* by al–Khwarizmi. Although it has now become a branch of mathematics that deals with structure, relation, and quantity, 5000 years ago it was confined to solving linear $ax = b$, quadratic $ax^2 + bx = c$, $ax^2 = bx + c$, $ax^2 + c = bx$, and indeterminate equations $ax + c = by$ and $x^2 + y^2 = z^2$, where a, b, and c are given positive integers. For linear and quadratic equations, the Indians in their Sulvasutras and Babylonians on their Cuneiform Tablets found exact (both determinate and indeterminate) positive solutions by the same method that we use today.

The Chinese mathematician Liu Hui (around 220–280), in his commentary on the *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art, which is believed to have been written around 1000 BC), reports in Chapter 7 a clever method (the rule of false position) for solving systems of two simultaneous linear equations with two unknowns, and in Chapter 8 he states a method (now known as Gaussian elimination) for finding the solutions of determinate and indeterminate systems of linear equations of higher order with both positive and negative numbers.

Until the 16th century algebra was mostly *rhetorical*, that is, each equation was expressed in ordinary language. The next stage of algebra, which was initiated by Brahmagupta (born 30 BC) and Diophantus of Alexandria (about 250) and continued by the Arabian mathematicians, has been called *syncopated*, and in it symbols are partially used. Finally, *symbolic* algebra, which we use today, became established through the works of Francois Viéte (1540–1603), René Descartes (1596–1650), and John Wallis (1616–1703). In the nineteenth century, symbols were used to write most mathematics as formulas. These formulas played a key role in advancing mathematics, particularly algebra, to great heights.

In appreciation of these formulas, Heinrich Hertz (1857–1892), the discoverer of electromagnetic waves, said "One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them".

In recent years various different types of algebra have emerged: abstract algebra (with topics such as groups, fields, and rings), associative algebra, Banach algebra, Boolean algebra, Borel algebra, C^*–algebra, Clifford algebra, commutative algebra, computer algebra, Heyting algebra, homological algebra, Hopf algebra, Lie algebra (the theory of continuous groups), relational algebra, sigma–algebra, symmetric algebra, tensor algebra, universal algebra, vector algebra, von Neumann algebra, and the list continues. Algebraists keep themselves busy by searching for new qualitative and quantitative results and their applications in diverse fields.

The word **Trigonometry** comes from two Greek words meaning 'triangle' and 'measure'. It deals with triangles and the relationship between the lengths of their sides and the angles between those sides. **Spherical Trigonometry** studies triangles on spherical polygons defined by a number of intersecting great circles on the sphere. In antiquity, trigonometry was studied as a branch of astronomy, and for this purpose several astronomical rules for spherical triangles were discovered that are scattered all over ancient astronomical texts such as Surya Siddhanta and its commentaries:

$$\begin{aligned}&\cos a = \cos b\cos c + \sin b\sin c\cos A\\ &\cos A\sin c = \cos a\cos b - \sin a\cos b\cos C\\ &\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C},\end{aligned}$$

where A, B, C are the angles of a spherical triangle, of which the opposite sides are a, b, and c, respectively.

Bhaskara I (before 123 BC) gave the following point–wise approximation formula for $\sin x$, which has a relative error less than 1.9%

$$\sin x \;=\; \frac{16x(\pi - x)}{[5\pi^2 - 4x(\pi - x)]}.$$

Daivajna Varahamihira (working 123 BC) gave trigonometric formulae that correspond to

$$\sin x = \cos(\pi/2 - x),\;\; \sin^2 x + \cos^2 x = 1 \quad \text{and} \quad (1 - \cos 2x)/2 = \sin^2 x.$$

Alexandrian Claudius Ptolemaeus (Ptolemy) (around 90–168 AD) derived the half–angle formula

$$\sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos x}{2}$$

and used it to construct trigonometric tables.

Ptolemy's thirteen books of the *Almagest* are the most influential and significant trigonometric work of all antiquity. Almost three centuries after the Almagest, Bhaskara II or Bhaskaracharya (working 486) developed spherical trigonometry systematically and discovered many trigonometric results, such as

$$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b.$$

Towards the end of the eighth century the knowledge of trigonometric functions and methods reached Persian and Arabic astronomers, who advanced it to its place as a major mathematical discipline independent of astronomy. We note that the formula

$$\sin 3x = 3\sin x - 4\sin^3 x$$

is due to Jemshid al–Kashi (around 1380–1429), which has been attributed to Viéte. In the fifteenth century, trigonometry reached Europe via Latin translations of the writings of these Persian and Arabic astronomers.

In antiquity, the general problem of finding area/volume of a geometric figure came to be known as the **quadrature problem**. In the Sulvasutras and the Cuneiform tablets we find formulas for the areas of squares, rectangles, right triangles, certain trapezoids, and the volumes of prisms, right circular cylinders, and the frustums of cones or square pyramids. Around 1850 BC, the Egyptians knew the correct formula for obtaining the volume of a frustum, that is,

$$V = \frac{1}{3}h(a^2 + ab + b^2);$$

here, h is the altitude, and a and b are the lengths of the sides of the square top and square base, respectively (when $a = 0$ this reduces to the well–known formula for the volume of the pyramid, $V = (1/3)hb^2$). This formula is a subject of debate even today.

Another important exact formula for finding the area A of a triangle having sides a, b, and c, known in the literature as Heron's (Heron of Alexandria, about 75 AD), was stated earlier by Bhaskara I:

$$A \;=\; \sqrt{s(s-a)(s-b)(s-c)},$$

where $2s = a + b + c$. This formula was later generalized by Brahmagupta for finding the area A of any cyclic quadrilateral given its four sides a, b, c, d as

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $2s = a + b + c + d$, which was later rediscovered by Willebrord Snell (1580–1626).

Antiphon of Rhamnos (around 480–411 BC) and Bryson of Heraclea (about 450 BC) tried to find the area of a circle by inscribing it by a regular polygon with an infinite number of sides. Around 240 BC, Archimedes of Syracuse (287–212 BC) used Aristotle's (around 384–322 BC) idea of the potential infinity to generalize *the method of exhaustion* of Eudoxus of Cnidus (around 400–347 BC), which formalized the ideas of Antiphon and Bryson. He used this method to find areas bounded by parabolas and spirals, the volumes of cylinders, areas of segments of spheres, and especially to approximate π by bounding its value between 22/7 and 223/71. Although Archimedes' geometric method (the circumference of a circle lies between the perimeters of the inscribed and circumscribed regular polygons of n sides, and as n increases, the deviation of the circumference from the two perimeters becomes smaller) to approximate the value of π is heuristic (not regarded as final and strict but merely as provisional and plausible), it foreshadowed the concept of the limit.

The quadrature problem eventually gave rise to **integral calculus**, which is the assimilation of the geometric and analytic methods and the understanding of *the calculus of infinitesimals* or *infinitesimal calculus*. This final step was taken independently by both Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716). The term 'integral' appeared for the first time in a paper produced by Jacob Bernoulli (1654–1705) in 1690, and the term 'integral calculus' was introduced by Leibniz and Johann Bernoulli (1667–1748) in 1698. Since this pioneering work by George Friedrich Bernhard Riemann (1826–1866) and Henri Léon Lebesgue (1875–1941), integral calculus has become a universal method for calculating area, volume, arc length, center of mass, work, and pressure.

The word **tangent** comes from the Latin 'tangere', to touch. The term **tangent line** is due to Leibniz, 1692, 1694, who defined it as the line through a pair of infinitely close points on the curve. In 1629, Fermat invented a general geometric procedure for drawing tangent lines, or simply, tangents on curves whose analytic forms were known. He made this great discovery while trying to find the maxima and minima of these curves. During 1664–1666, Isaac Barrow (1630–1677) developed a new geometric method for determining tangents on curves. In fact, in the 17th century, the tangent line problem became one of the central questions in geometry. Descartes remarked

"And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know".

In Europe, the tangent problem prefigured **differential calculus**, another term that was coined by Leibniz in 1684, which has been credited solely to Newton and Leibniz. For Newton, the calculus was geometrical, while Leibniz took it toward analysis. However, the concept of a derivative had been developed almost 1200 years before Newton and Leibniz by Bhaskara II, who provided differentiation of the trigonometric functions, for example (in modern notation), he established $\delta(\sin x) = \cos x \cdot \delta x$. Bhaskara II also gave a statement of Michel Rolle's (1652–1719) theorem, concluded that the derivative vanishes at a maxima, and introduced the concept of the instantaneous motion of a planet in his collection *Siddhanta Siromani.* Then, in the 14th century, Madhava of Sangamagramma (1340–1425) invented the ideas underlying the infinite series expansions of functions, power series, the trigonometric series of sine, cosine, tangent, and arctangent (these series have been credited to James Gregory (1638–1675), Brook Taylor (1685–1731), and Newton), rational approximations of infinite series, tests of convergence of infinite series, the estimate of an error term, and early forms of differentiation and integration.

Madhava fully understood the limit nature of the infinite series. This step has been called the "decisive factor onward from the finite procedures of ancient mathematics to treat their limit–passage to infinity", which is in fact the kernel of modern classical analysis. Parameshvara Namboodri (around 1370–1460), a disciple of Madhava, stated an early version of the mean value theorem in his *Lilavathi Bhasya.* This is considered to be one of the most important results in differential calculus, and was later essential in proving the fundamental theorem of calculus, which shows the inverse character of tangent and area problems. Evangelista Torricelli (1608–1647) was the first to understand the fundamental theorem of calculus geometrically, and this was extended by Gregory while Barrow established a more generalized version, and finally Newton completed the mathematical theory.

Nilakanthan Somayaji (around 1444–1544), following the footsteps of Madhava and his father Parameshvara, provided a derivation and proof of the arctangent trigonometric series and gave the relationship between the power series of π and arctangent, namely,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots,$$

which in the literature has been credited to Gregory and Leibniz. Guillaume Francois Antoine de L'Hôpital (1661–1704) is known as the author of the world's first text book on differential calculus, but Jyesthadevan (around 1500–1600) wrote the calculus text *Yuktibhasa* in Malayalam (a regional language of the Indian state of Kerala) almost 150 years earlier.

The work of Leibniz, the Bernoulli brothers, Euler, Bernhard Placidus Johann Nepomuk Bolzano (1781–1848), Augustin–Louis Cauchy (1789–1857), Karl Theodor Wilhelm Weierstrass (1815–1897), Riemann, and several other 18th and 19th century mathematicians made calculus one of the most powerful, supple, and practical tools of mathematics. Acknowledging the importance of calculus, John Louis von Neumann (1903–1957) said that

"the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking".

Today calculus, its advances and abstractions, and its applied branches such as differential equations, optimization, etc. have become a major part of mathematical education at all levels.

In mathematics a **set** is defined as a collection of objects or elements, and **set theory** is considered to be a branch of mathematical logic. It has been documented that Georg Cantor (1845–1918) introduced set theory to the mathematical world in 1874. However, during 200–875 AD, the Jain School of Mathematics in India utilized the concept of sets. In their work, the Jains introduced several different types of sets, such as cosmological, philosophical, karmic, finite, infinite, transfinite, and variable sets. They called the largest set an omniscient set, and the set containing no elements was known as the null set. They also defined the concept of a union of sets and used the method of one–to–one correspondence for the comparison of transfinite sets. In order to determine the order of comparability of all sets, they considered fourteen types of monotone sequences. Thus, Cantor alone is not the founder of set theory. His major contribution was the mathematical systematization of set theory, now known as naive (non–axiomatic) set theory, and the modern understanding of infinity.

After Cantor's work, naive set theory became widespread; however, by 1908 it led to several paradoxes (also known as antinomies), the simplest and best known of which is Bertrand Arthur William Russell's (1872–1970) paradox. This gave rise to numerous attempts at finding a solution, and finally resulted in the three main philosophies or schools of thought, namely, the logicist, formalist, and intuitionist schools.

Today, set theory is woven into the fabric of modern mathematics. In fact, the language of set theory is used to precisely define nearly all mathematical objects. It has lead to many interrelated subfields (such as combinatorial set theory, descriptive set theory, fuzzy set theory, set–theoretic topology, computable set theory, category theory, etc.) which are most active fields of mathematical research. One of the most exciting aspects of set theory is that its elements can be studied informally and intuitively, and so it is being taught in primary schools using John Venn (1834–1923) diagrams.

After Pythagoras several different definitions of mathematics have been proposed. Each one tries to define mathematics with a specific context in mind. For example:

Mathematics is like draughts (checkers) in being suitable for the young, not too difficult, amusing, and without peril of the state. (Plato)

Mathematics is the study of "quantity". (Aristotle)

Mathematics is the door and key to the sciences. (Roger Bacon, 1214–1294)

Mathematics is the science of order and measure. (Descartes)

Mathematics–the unshaken foundation of science, and the plentiful fountain of advantage to human affairs. (Barrow)

Mathematics is concerned only with the enumeration and comparison of relations. (Gauss)

Mathematics is the science of what is clear by itself. (Carl Guslov Jacob Jacobi, 1804–1851)

Mathematics is the science which draws necessary conclusions. (Benjamin Peirce, 1809–1880)

Mathematics seems to endow one with something like a new sense. (Charles Robert Darwin, 1809–1882)

Mathematics is the work of the human mind, which is destined rather to study than to know, to seek the truth rather than to find it. (Évariste Galöis, 1811–1832)

Mathematics is not (as some dictionaries today still assert) merely "the science of measurement and number", but, more broadly, any study consisting of symbols along with precise rules of operation upon those symbols, the rules being subject only to the requirement of inner consistency. (George Boole, 1815–1864)

"Do you know what a mathematician is" Lord Kelvin (1824–1907) once asked a class. He stepped to the board and wrote

$$\int_{-\infty}^{\infty} e^{-x^2} dx \ = \ \sqrt{\pi}.$$

Putting his finger on what he had written, he turned to the class. "A mathematician is one to whom that is as obvious as that twice two makes four is to you".

Mathematics is a branch of logic. (Julius Wilhelm Richard Dedekind, 1831–1916)

Mathematics is the science of quantity. (Charles Sanders Peirce, 1839–1914)

Mathematics is the science of self–evident things. (Felix Christian Klein, 1849–1925)

Mathematics is the development "of all types of formal, necessary and deductive reasoning". (Alfred North Whitehead, 1861–1947)

Mathematics is a game played according to certain rules with meaningless marks on paper. (David Hilbert, 1862–1943)

Mathematics is a subject identical with logic. (Russell)

The whole of mathematics is nothing more than a refinement of everyday thinking. (Einstein)

Mathematics is the art of problem solving. (George Polya, 1887–1985)

Mathematics is a development of thought that had its beginning with the origin of man and culture a million years or so ago. To be sure, little progress was made during hundreds of thousands of years. (Leslie Alvin White, 1900–1975)

Mathematics is a spirit of rationality. It is this spirit that challenges, simulates, invigorates and drives human minds to exercise themselves to the fullest. It is this spirit that seeks to influence decisively the physical, normal and social life of man, that seeks to answer the problems posed by our very existence, that strives to understand and control nature and that exerts itself to explore and establish the deepest and utmost implications of knowledge already obtained. (Morris Kline, 1908–1992)

Mathematics today is the instrument by which the subtle and new phenomena of nature that we are discovering can be understood and coordinated into a unified whole. In this some of the most advanced and newest branches of mathematics have to be employed and contact with an active school of mathematics is therefore great asset for theoretical physicists. (Homi Jehangir Bhabha, FRS, 1909–1966)

Mathematics is like a chest of tools, before studying the tools in detail, a good workman should know the object of each, when it is used and what it is used for. (Walter Warwick Sawyer, 1911–2008)

Mathematics has also been defined as follows: mathematics is a human creation; mathematics is a natural part of man's cultural heritage; mathematics is the accumulation of human wisdom in an effort to understand and harness the physical, social, and economic worlds; mathematics is a language and a language can be learned only by continuously using it; mathematics is a tool that ideally permits mediocre minds to solve complicated problems expeditiously; mathematics is the study of quantity, structure, space, and change; mathematics is the measurement, properties, and relationships of quantities and sets using numbers and symbols; mathematics is something that man himself creates, and the type of mathematics he works out is just as much a function of the cultural demands of the time as any of his other adaptive mechanisms; mathematics is the science which uses easy words for hard ideas; and, mathematics seeks regularities and pattern in behavior, motion, number, or shape, or even in the substrata of chaos.

A variety of quips and clichés can also define mathematics: "It's an art", "it's a science – in fact, it's the queen and servant of science", "it's what I use when I balance my checkbook", "a game that we play with rules we're not quite sure of", and the apologetic favorite "something I was never good at", and the list can go on and on.

About mathematics several interesting positive views also have been proposed. Perhaps from these we can find some more definitions of mathematics.

Mathematics gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas, she abolishes the oblivion and ignorance which are ours by birth. (Proclus Diadochus, 410–485 AD)

Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. (Roger Bacon)

Nature is written in mathematical language. (Galileo Galilei, 1564–1642)

Mathematicians are like Frenchmen; whatever you say to them they translate into their own language, and forthwith it is something entirely different. (Johann Wolfgang von Goethe, 1749–1832)

Mathematics is one of the oldest of sciences; it is also one of the most active; for its strength is the vigor of perpetual youth. (Andrew Russell Forsyth, 1858–1942)

In mathematics I can report no deficiency, except it be that men do not sufficiently understand the excellent use of the Pure Mathematics. (Francis Bacon, 1909–1992)

Yet another positive view of mathematics and hence of mathematicians follows. Mathematicians are of two types: the 'we can' men believe (possibly subconsciously) that mathematics is a purely human invention; the 'there exists' men believe that mathematics has an extra–human *existence* of its own, and that 'we' merely come upon the *eternal truths* of mathematics in our journey through life, in much the same way that a man taking a walk in a city comes across a number of streets with whose planning he had nothing whatever to do (Bell, Men of Mathematics).

We also encounter some negative views about mathematics, such as the idea that the mathematics that is certain does not refer to reality and mathematics that refers to reality is not certain. Sir William Hamilton (1788-1856), the famed Scottish philosopher, logician, and meta–physicist, viewed mathematics in a way that may be construed as a cruel attack on mathematics and hence on mathematicians: "Mathematics freeze and parch the mind", "an excessive study of mathematics absolutely incapacitates the mind for those intellectual energies which philosophy and life require", "mathematics cannot conduce to logical habits at all", "in mathematics dullness is thus elevated into talent, and talent degraded into incapacity", "mathematics may distort, but can never rectify, the mind". (Bell, Men of Mathematics).

The following parable of "The Blind Men and the Elephant" (a Story from the *Buddhist Sutra*) is relevant to our attempt to define mathematics. Several prominent citizens were engaged in a hot argument about God and the different religions, and could not come to an agreement. So they approached Lord Gautama Buddha (1887–1807 BC) to find out what exactly God looks like. Buddha asked his disciple to get a large majestic elephant and four blind men. He then brought the four blind men to the elephant and told them to find out what the elephant would "look" like. The first blind man touched the elephant's leg and reported that it "looked" like a pillar.

The second blind man touched its tummy and said that an elephant was an inverted ceiling. The third blind man touched the elephant's ear and said that it was a piece of cloth. The fourth blind man held on to the tail and described the elephant as a piece of rope. And all of them ran into a hot argument about the "appearance" of an elephant. The Buddha asked the citizens: "Each blind man had touched the elephant but each of them gives a different description of the animal. Which answer is right?" "All of them are right," was the reply. "Why? Because everyone can only see part of the elephant. They are not able to see the whole animal. The same applies to God and to religions. No one will see Him completely." By this story, Lord Buddha teaches that we should respect all legitimate religions and their beliefs.

This famous "blind men" episode is not meant to disrespect any mathematician. We state it only to point out that *any definition of mathematics, however elaborate or epigrammatic, will fail to lay bare its fundamental structure and the reasons for its universality* (Hermann Klaus Hugo Weyl, 1885–1955). In fact, the definition of mathematics continues to change with time and innovation. However, mathematics rewards its creator with a strong sense of aesthetic satisfaction. It helps us understand man's place in the universe and enables us to find order in chaos. Under certain axioms, mathematics is the most absolute, ever lasting, precise, significant, and universal subject. It is perceived as the highest form of thought in the world of learning. No doubt, mathematics is one of the greatest creations of mankind–if it is not indeed the greatest. Mathematics will live forever.

Finally, I will like to mention the work of the following two Romanian mathematicians:

Traian Lalescu (July 12, 1882 June 15, 1929). His main focus was on integral equations and he contributed to work in the areas of functional equations, trigonometric series, mathematical physics, geometry, mechanics, algebra, and the history of mathematics. He wrote world's first book on Integral Equations in 1912-one year before Vito Volterra's book, namely, T. Lalesco, Introduction á la théorie des équations intégrales. (avec une préface de É. Picard), A. Hermann et Fils, Paris, 1912. VII + 152 pp.

Dimitrie Pompeiu (October 4, 1873 October 8, 1954) made important contributions to mathematical analysis, complex functions theory, and rational mechanics. In 1906, he constructed a nonconstant, everywhere differentiable functions, with derivative vanishing on a dense set. Such derivatives are now called Pompeiu's derivatives. Another important idea of Pompeiu's doctoral thesis (1905) was the distance between two sets which he called the "ecart". The distance between sets is now called the "Hausdorff metric"; however, Hausdorff's definition is slightly different, but both the definitions lead to the same topology. Thus, Pompeiu is considered as one of the founders of the theory of hyperspaces.