On the Existence of Isotone Galois Connections between Preorders

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Overline

1. Introduction and preliminaries.
2. Definitions in a preordering setting.
3. Building adjunctions
4. Conclusions and future work

Preliminary definitions and notations

Let $\mathbb{A} = (A, \leq_A)$ be a partially ordered set, $X \subseteq A$, and $a \in A$.

- **Upper bounds** of X:

$$UB(X) = \{u \in A \mid x \leq_A u \text{ for all } x \in X\}$$

- **Maximum** of X:

$$\max(X) = a \;\text{ iff }\; a \in UB(X) \cap X$$

- **Downward closure** of a:

$$a^{\downarrow} = \{x \in A \mid x \leq_A a\}$$

- **Upward closure** of a:

$$a^{\uparrow} = \{x \in A \mid a \leq_A x\}$$

Preliminary definitions and notations

A mapping $f\colon (A, \leq_A) \to (B, \leq_B)$ between partially ordered sets is said to be

- **isotone** if, for all $a_1, a_2 \in A$,

$$a_1 \leq_A a_2 \;\text{ implies }\; f(a_1) \leq_B f(a_2)$$

- **antitone** if, for all $a_1, a_2 \in A$,

$$a_1 \leq_A a_2 \;\text{ implies }\; f(a_2) \leq_B f(a_1)$$

In the particular case in which $A = B$,

- f is **inflationary** (also called extensive) if, for all $a \in A$,

$$a \leq_A f(a)$$

- f is **deflationary** if, for all $a \in A$,

$$f(a) \leq_A a$$

The definition of Adjunction

Let $\mathbb{A} = (A, \leq_A)$ and $\mathbb{B} = (B, \leq_B)$ be posets, and $f : A \to B$ and $g : B \to A$ be two mappings.

The pair (f, g) is said to be an **adjunction** or *isotone Galois connection between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by

$$(f, g) \colon \mathbb{A} \leftrightharpoons \mathbb{B}$$

if, for all $a \in A$ and $b \in B$, the following condition holds

$$f(a) \leq_B b \qquad \text{if and only if} \qquad a \leq_A g(b)$$

The mapping f is called **left adjoint** and g is called **right adjoint**.

Basic definitions on preordered sets

A preordered set is a pair $(A, \lesssim_A)$ where $\lesssim_A$ is a reflexive and transitive binary relation on A.

Definition

Given a preordered set $(A, \lesssim_A)$ and a subset $X \subseteq A$,

- Set of **p-maximum** elements of X:

$$\mathrm{p\text{-}max}(X) = \{a \in X \mid x \lesssim_A a \text{ for all } x \in X\}$$

- Set of **p-minimum** elements of X

$$\mathrm{p\text{-}min}(X) = \{a \in X \mid a \lesssim_A x \text{ for all } x \in X\}$$

Notice that $\mathrm{p\text{-}max}(X)$ (resp., $\mathrm{p\text{-}min}(X)$) need not be a singleton because of the absence of antisymmetry.

Characterization of adjunctions

Theorem

Let $\mathbb{A} = (A, \lesssim_A)$ and $\mathbb{B} = (B, \lesssim_B)$ be two preordered sets, and $f : A \to B$ and $g : B \to A$ be two mappings.
The following statements are equivalent:

1. $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.
2. *f and g are isotone maps,*
   $g \circ f$ is inflationary, and $f \circ g$ is deflationary.
3. $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ *for all* $a \in A$.
4. $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ *for all* $b \in B$.
5. *f is isotone and* $g(b) \in \mathrm{p\text{-}max}\, f^{-1}(b^{\downarrow})$ *for all* $b \in B$.
6. *g is isotone and* $f(a) \in \mathrm{p\text{-}min}\, g^{-1}(a^{\uparrow})$ *for all* $a \in A$.

P-kernel relation

- Let $\mathbb{A} = (A, \lesssim_A)$ be a preordered set. The **symmetric kernel** is the equivalence relation $\approx_A$ defined as follows: for $a_1, a_2 \in A$,

$$a_1 \approx_A a_2 \quad \text{if and only if} \quad a_1 \lesssim_A a_2 \text{ and } a_2 \lesssim_A a_1$$

- Given a mapping $f \colon A \to B$ the **kernel** relation $\equiv_f$ is defined as follows: for $a_1, a_2 \in A$,

$$a_1 \equiv_f a_2 \quad \text{if and only if} \quad f(a_1) = f(a_2)$$

P-kernel relation

The **p-kernel** relation $\cong_A$ is the equivalence relation obtained as the transitive closure of the union of the relations $\approx_A$ and $\equiv_f$.

$$\cong_A \;=\; (\approx_A \cup \equiv_f)^{tr}$$

Hoare preorder

Definition

Let $(A, \lesssim_A)$ be a preordered set, and consider $X, Y \subseteq A$.

$$X \sqsubseteq Y \text{ iff for all } x \in X, \text{ there exists } y \in Y \text{ such that } x \leq y.$$

Lemma

Let $(A, \lesssim)$ be a preordered set, and consider $X, Y \subseteq A$ such that $\mathrm{p\text{-}min}(X) \neq \varnothing$ and $\mathrm{p\text{-}min}(Y) \neq \varnothing$.
The following statements are equivalent:

1. $\mathrm{p\text{-}min}(X) \sqsubseteq \mathrm{p\text{-}min}(Y)$
2. *For all $x \in \mathrm{p\text{-}min}(X)$ and for all $y \in \mathrm{p\text{-}min}(Y)$, $x \leq y$.*

Building adjunctions on posets

Theorem (García et al, IPMU14)

Let $(A, \leq_A)$ be a poset and $f\colon A \to B$.
There exist an ordering $\leq_B$ in B and a mapping $g\colon B \to A$ such that $(f, g)\colon A \leftrightharpoons B$ if and only if

1. *There exists $\max([a]_{\equiv_f})$ for all $a \in A$.*
2. *$a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.*

Conditions for the existence of an adjunction

Let $\mathbb{A} = (A, \lesssim_A)$ and $\mathbb{B} = (B, \lesssim_B)$ be two preordered sets and let $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$. Consider the set $S = g(f(A))$.

Then, the following conditions hold:

1. $g(f(a)) \in \mathrm{p}\text{-max}[g(f(a))]_{\cong_A}$, for all $a \in A$.
2. $g(f(a)) \in \mathrm{p}\text{-min}(UB[a]_{\cong_A} \cap S)$, for all $a \in A$.
3. If $a_1 \lesssim_A a_2$, then $\mathrm{p}\text{-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \mathrm{p}\text{-min}(UB[a_2]_{\cong_A} \cap S)$.

Sufficient conditions to build a right adjoint

Lemma

Let $\mathbb{A} = (A, \lesssim_A)$ be a preordered set and $f\colon A \to B$ be an **onto** *map. Let $S \subseteq A$ such that the following conditions hold:*

1. $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$, *for all $a \in A$.*
3. *If $a_1 \leq_A a_2$, then* $\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)$

Then, there exist a preordering $\lesssim_B$ in B and a map $g\colon B \to A$ such that $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.

The construction

- Under the previous hypotheses, the preordering relation in B is defined as follows:

$$b_1 \lesssim_B b_2 \text{ if and only if}$$

$$\text{there exist } a_1 \in f^{-1}(b_1) \text{ and } a_2 \in f^{-1}(b_2) \text{ such that}$$

$$\mathrm{p\text{-}min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \mathrm{p\text{-}min}(UB[a_2]_{\cong_A} \cap S).$$

- The definition of $g \colon B \to A$ is not unique, because all the functions such that, for all $b \in B$,

$$g(b) \in \mathrm{p\text{-}min}(UB[x_b]_{\cong_A} \cap S) \quad \text{being } x_b \in f^{-1}(b)$$

are suitable to define the adjunction.

Extension from the image set to whole codomain

Consider $(A, \lesssim_A)$ a preordered set, B a set, and $f\colon A \to B$.
If there exists an adjunction $(f, g')\colon (A, \lesssim_A) \leftrightharpoons (f(A), \lesssim_{f(A)})$, then, there exist both a preorder $\lesssim_B$ on B and an adjunction

$$(f, g)\colon (A, \lesssim_A) \leftrightharpoons (B, \lesssim_B)$$

Fix $m \in f(A)$ and choose $\lesssim_B$ to be the reflexive and transitive closure of the relation $\lesssim_{f(A)} \cup \{(m, y) \mid y \notin f(A)\}$ and

$$g(x) = \begin{cases} g'(x) & \text{if } x \in f(A) \\ g'(m) & \text{if } x \notin f(A) \end{cases}$$

Main contribution

Theorem

Let $\mathbb{A} = (A, \lesssim_A)$ *be a preordered set,* $f\colon A \to B$ *be a mapping.*

Then, there exist a preorder $\mathbb{B} = (B, \lesssim_B)$ *and* $g\colon B \to A$ *such that* $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$

if and only if

there exists $S \subseteq A$ *such that*

1. $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$*, for all* $a \in A$.
3. *If* $a_1 \leq_A a_2$*, then* $\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)$.

Conclusions

- We have studied the existence and construction of the adjoint pair to a given mapping f, but in the more general framework of preordered sets.
- The absence of antisymmetry makes both the statements and the proofs of the results to be much more involved than in the ordered setting.
- Contrariwise to the partially ordered case, given a preordered set $\mathbb{A} = (A, \lesssim_A)$ and an onto mapping $f \colon A \to B$, the unicity of neither the preordering $\lesssim_B$ nor the mapping $g \colon B \to A$ satisfying $(f, g) \colon \mathbb{A} \leftrightharpoons \mathbb{B}$, when it exists, can be guaranteed.

Future work

- Alternative approaches to this problem in order to obtain, if possible, a simpler alternative characterization.
- Possible applications to generalizations in FCA which weaken the structure on which a Galois connection is defined and to knowledge discovery.
- Extending the results to a fuzzy setting.

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