Automatized Construction of Implicative Theory of Algebraic Identities of Size up to 5

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Formal Contexts & Implications

Definition (Formal Context)

M - a set of attributes.
G - a set of objects.
I - a relation between G and M.
$\mathbb{K} = (G, M, I)$ - a (formal) context.

	all sides equal	some sides equal	has right angle
◊	x	x	
□	x	x	x
▭		x	x
▱		x	

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$\{\text{all sides equal}\} \rightarrow \text{some sides equal}$

(Unit) Implication $Y \rightarrow z$, $Y \subseteq M, z \in M$

$\forall g \in G :$ if gIY then gIz.

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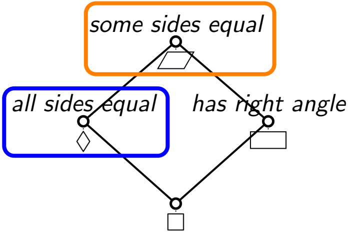

Implication Base

Implication base of context

Set of implications:

- from which any valid implication can be deduced,
- none of the proper subsets has this property.

Example

1. all sides equal $\to$ some sides equal;
2. has right angle $\to$ some sides equal.

Implication Base

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- from which any valid implication can be deduced,
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Example

1. all sides equal $\rightarrow$ some sides equal;
2. has right angle $\rightarrow$ some sides equal.

Counter-example

	all sides equal	some sides equal	has right angle
			x

Attribute Exploration of Quadrangles

	all sides equal	some sides equal	has right angle	all angles equal
	×	×		
	×	×	×	×
		×	×	×
		×		
				

1. all sides equal $\rightarrow$ some sides equal;
2. all angles equal $\rightarrow$ some sides equal, has right angle;
3. has right angle $\rightarrow$ some sides equal, all angles equal.

Attribute Exploration of Quadrangles

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		×		
				
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Identities

Operations Φ:

binary: $f^{(2)}$ or $*$

unary: $f^{(1)}$ or $-$

nullary: $f^{(0)}$ or a

Variables X:

x, y, z

Terms $T_\Phi(X)$

Terms construction:

- $X \subseteq T_\Phi(X)$;
- $p_1, \ldots p_n \in T_\Phi(X)$ and $f^{(n)} \Rightarrow f^{(n)}(p_1, \ldots p_n) \in T_\Phi(X)$.

Identity

Identity is a pair (p, q), $p, q \in T_\Phi(X)$, written $p \equiv q$.

Size of Identity

Size of identity $p_1 \equiv p_2$

$$\begin{array}{l} v(p_i) \\ o(p_i) \end{array} \text{ number of occurrences of } \begin{array}{r} \text{variables} \\ \text{operations} \end{array} \text{ in } p_i.$$

Size: $l(p_1 \equiv p_2) = \sum_{i=1}^{2} v(p_i) + o(p_i)$.

Example ($x * y \equiv y * x$)

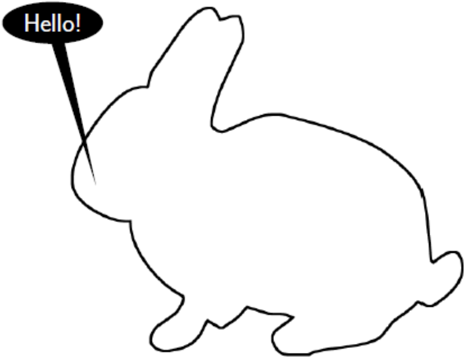
$v(x * y) = 2,$ $v(y * x) = 2,$

$o(x * y) = 1,$ $o(y * x) = 1,$

$$l(x * y \equiv y * x) = 2 + 1 + 2 + 1 = 6.$$

It is necessary to limit the size.

BUNny



BUNny

B	U	N	n	y
i	n	u		
n	a	l		
a	r	l		
r	y	a		
y		r		
		y		

Example (A bunny of size 2)

$\mathfrak{B}_5 = (\{0,1\}, (*, -, 0))$

*	0	1
0	0	0
1	1	0

-	0	1
	0	0

Definition (BUNny)

A set and a family of a binary, a unary, and a nullary operations are called a bunny.

Equivalent Identities

Satisfaction

$p^{(n)}(\bar{x}) \equiv q^{(n)}(\bar{x})$ is satisfied in $(A, (f^{(2)}, f^{(1)}, f^{(0)}))$ iff $p^{(n)}(\bar{a}) = q^{(n)}(\bar{a})$ for all $\bar{a} \in A^n$.

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Equivalent identities $p_1 \equiv p_2 \iff q_1 \equiv q_2$

Identities are equivalent iff they are satisfied in the same bunnies.

Examples

- $a \equiv a \iff x \equiv x$,

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Identities are equivalent iff they are satisfied in the same bunnies.

Examples

- $a \equiv a \iff x \equiv x$,
- $x \equiv y \iff x \equiv a$,
- $x \equiv a \iff x \equiv -y$.

It makes sense to avoid equivalent identities.

Motivation

Definition (Equational Class)

A class of algebras satisfying given identities.

Example (Semigroup)

The equational class of algebras with only a binary operation satisfying

$$x * (y * z) \equiv (x * y) * z;$$

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Example (Group)

The equational class of algebras with a binary, a unary, and a nullary operations satisfying

$$x * (y * z) \equiv (x * y) * z;$$
$$x * 1 \equiv 1 * x \equiv x;$$
$$x * (-x) \equiv (-x) * x \equiv 1;$$

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Example (Abelian Group)

The equational class of algebras with a binary, a unary, and a nullary operations satisfying

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$$x * (-x) \equiv (-x) * x \equiv 1;$$
$$x * y \equiv y * x.$$

Motivation

Definition (Equational Class)

A class of algebras satisfying given identities.

Definition (Variety)

A class of algebras closed under homomorphisms, subalgebras, products.

Theorem (HSP, Birkhoff 1935)

$$\textit{Variety} \quad \sim \quad \textit{equational class.}$$

Problem Statement

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Automatic construction of the implicative theory of algebraic identities of size up to 5 (70 pairwise non-equivalent).

Related research

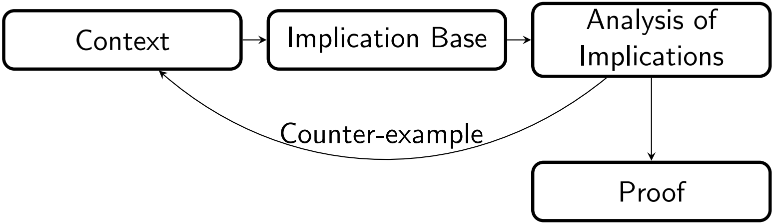
Concerning:

- Decidability of equational theories [Per67], [Tay79];
- Finding (finite) bases [BS81].

Closely related work

PhD thesis, most of research by hand [Kes13].

Attribute Exploration of Equational Classes



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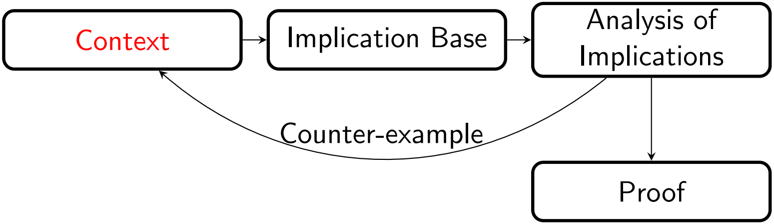
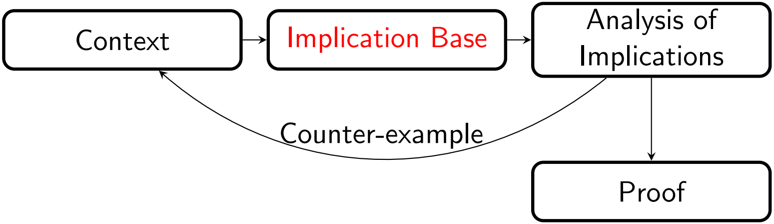


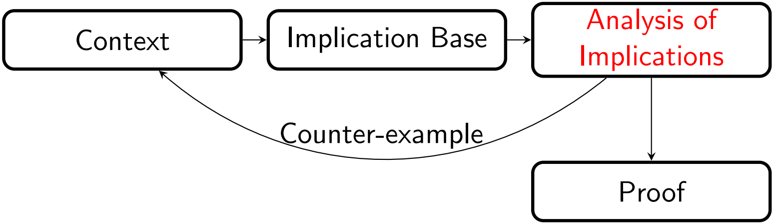
- On the first step initial (small) context is generated;
- On every step unnecessary objects are eliminated.

Attribute Exploration of Equational Classes



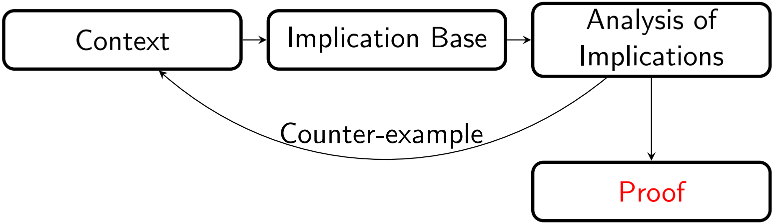
- Canonical base.

Attribute Exploration of Equational Classes



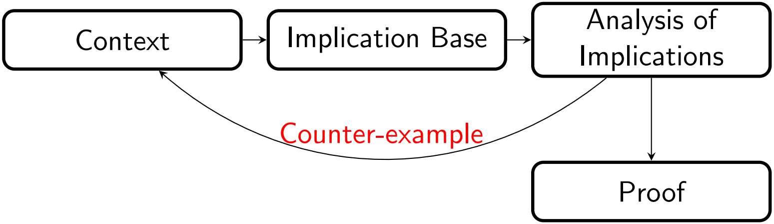
- Try to prove;
- Try to find a counter-example.

Attribute Exploration of Equational Classes



- Prover9 (from `http://www.cs.unm.edu/~mccune/mace4/`).

Attribute Exploration of Equational Classes



- Mace4 (from `http://www.cs.unm.edu/~mccune/mace4/`);
- `find_infinite_algebra`.

Only finite bunnies?

Lemma ([Kes13])

For finite bunnies is satisfied: $\{x \equiv a * (-x)\} \rightarrow \ x \equiv -(a * x)$.

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Counter-example: algebra $\mathcal{A}_\infty = (\mathbb{N}_0, (*_\infty, -_\infty, a_\infty))$

$$m *_\infty n = \begin{cases} n, & \text{if } m = 0 \text{ and } n \leq 2; \\ n -_{\mathbb{N}_0} 1, & \text{if } m = 0 \text{ and } n \geq 3; \\ 0, & \text{if } m \geq 1. \end{cases}$$

$$-_\infty n = \begin{cases} n, & \text{if } n \leq 2; \\ n + 1, & \text{if } n \geq 3. \end{cases}$$

$$a_\infty = 0.$$

Try $x = 3$.

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$$3 \equiv a * (-3) \quad \rightarrow \quad 3 \equiv -(a * 3).$$

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$$-_\infty n = \begin{cases} n, & \text{if } n \leq 2; \\ n + 1, & \text{if } n \geq 3. \end{cases}$$

$$a_\infty = 0.$$

$$3 \equiv 0 * 4 \quad \to \quad 3 \equiv -2.$$

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$$-_\infty n = \begin{cases} n, & \text{if } n \leq 2; \\ n + 1, & \text{if } n \geq 3. \end{cases}$$

$$a_\infty = 0.$$

$$3 \equiv 3 \quad \to \quad 3 \not\equiv 2.$$

Template

Simplified infinite bunny template

$$\mathfrak{B}_{inf} = (\mathbb{N}, (*, -, 0))$$

*	0	1	...
0	$\mathbb{N}_{\leq 3}$	$\mathbb{N}_{\leq 3}$	$case_3$
1	$\mathbb{N}_{\leq 3}$	$\mathbb{N}_{\leq 3}$	$case_4$
...	$case_1$	$case_2$	$case_5$

$case_i(x, y) = c_{i1} \times x + c_{i2} \times y + c_{i3}$, $c_{ij} \in [-3, 3]$

-	0	1	...
	$\mathbb{N}_{\leq 3}$	$\mathbb{N}_{\leq 3}$	$case$

$case(x) = d_1 \times x + d_2$, $d_i \in [-3, 3]$

Algorithm 1: find_infinite_algebra

Input: $P_{\text{ids}} \rightarrow C_{\text{id}}$, where $P_{\text{ids}} \subseteq M_{\text{id}}$, $C_{\text{id}} \in M_{\text{id}}$.
Output: Algebra $\mathcal{A} = (\mathcal{N}_0, (*, -, 0))$ satisfying all P_{ids} and not satisfying C_{id}.

while *True* **do**
    for *id* **in** P_{ids} **do**
        sat, term = check_identity($\mathcal{A}$, id)
        if *sat = False* **then**
            backtrack($\mathcal{A}$)
            break
        if *sat = None* **then**
            update($\mathcal{A}$, term)
            break
    else
        sat, term = check_identity($\mathcal{A}$, C_{id})
        if *sat = True* **then**
            backtrack($\mathcal{A}$)
            continue
        if *sat = None* **then**
            update($\mathcal{A}$, term)
            continue
        if *sat = False* **then**
            return $\mathcal{A}$

Results

Total number of bunnies found during exploration	27266
Number of finite bunnies	626
Number of infinite bunnies	1529
Number of (unit) implications	4398
Time taken	$\simeq$78 hours

Conclusion

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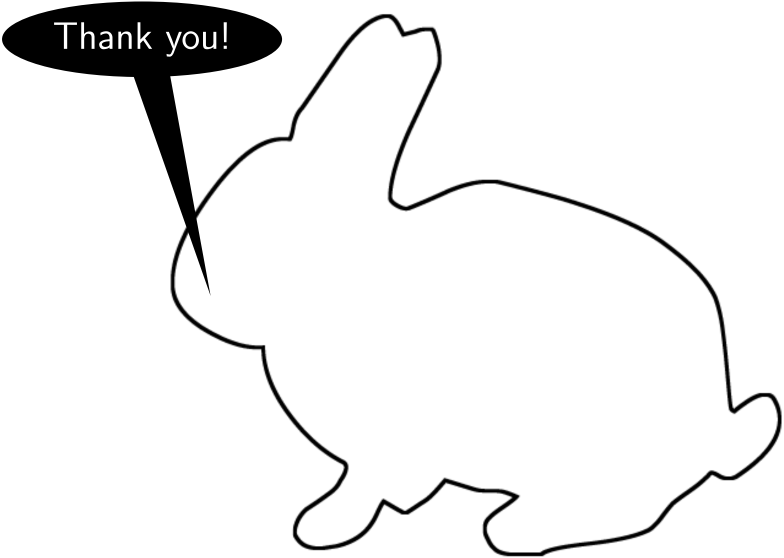
- Attribute exploration yields automatic procedure for accomplishing implication theory of equational classes;
- Introduced template suffices for considered identities.

Future work

- Can automatic Attribute Exploration be used in other domains?
- How is it possible to extend the template for bigger identities?



https://github.com/artreven

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