Learning Spaces, and How to Build them

JEAN-PAUL DOIGNON

Université Libre de Bruxelles

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Assessment in Computer-aided Education

A project initiated in 1984 with Jean-Claude Falmagne (NYU; UCI):

- an on-line teaching system which relies on the actual state of knowledge of the student;
- the assessment of knowledge needs to produce more than just a mark;
- in 2013, the software ALEKS was used by $1,300,000$ students (mostly in the US, half of them in higher education).

I will explain part of the deterministic theory, involving

"learning spaces" or "antimatroids",

with the focus on how to build a learning space in practice;

"semilattices" will play a rôle!

Not presented here:

- the assessment procedures;
- the probabilistic extension of the theory.

An Example of a Test

A test consists of a list of items (questions, problems) to be responded, as in the hypothetical example:

(a) $4 + 5 = ?$
(b) $34.6 \times 78.45 = ?$
(c) What is the area of a 4.2 by 9.4 rectangle?
(d) $\ldots$
(e) $\int_2^6 \frac{x^2-1}{x+2}\, dx = ?$

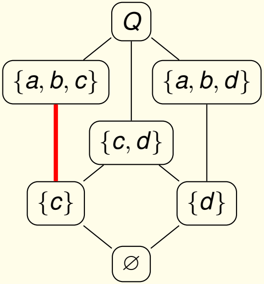
We identify what the student masters with the subset of items s/he correctly answered. Thus here item = notion.

The structure of a domain of knowledge consists of a collection of subsets, the potential 'knowledge states' of examinees.

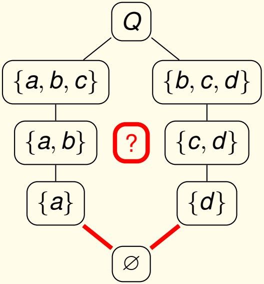
Two Inadequate Examples of Structures

The domain is $Q = \{a, b, c, d\}$, the states are shown:

$\mathcal{K}^{(1)}$



$\mathcal{K}^{(2)}$



In the talk, Q will always denote a finite set of items,
$\mathcal{K}$ and $\mathcal{L}$ collections of subsets of Q (that is: $\mathcal{K}, \mathcal{L} \subseteq 2^Q$).

Learning Spaces, Knowledge Spaces (1/3)

Definition

A **learning space** $(Q, \mathcal{L})$ consists of

a **domain** Q, which is a finite set;

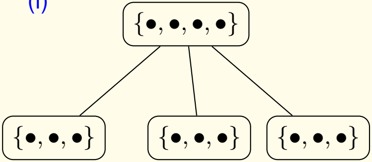
a collection $\mathcal{L}$ of subsets of Q, called the **(knowledge) states**, s.t.:

(i) $\mathcal{L}$ is **downgradable**: $\forall L \in \mathcal{L} \setminus \{\varnothing\},\ \exists q \in L : \ L \setminus \{q\} \in \mathcal{L}$;

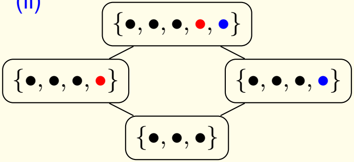
(ii) $\forall q, r \in Q,\ \forall L \in \mathcal{L} : \ \big(L \cup \{q\},\, L \cup \{r\} \in \mathcal{L}\big) \implies L \cup \{q, r\} \in \mathcal{L}$;

(iii) $Q \in \mathcal{L}$.

(i)

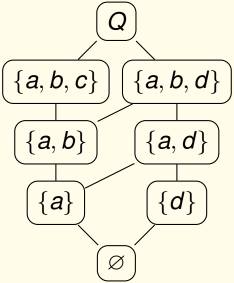


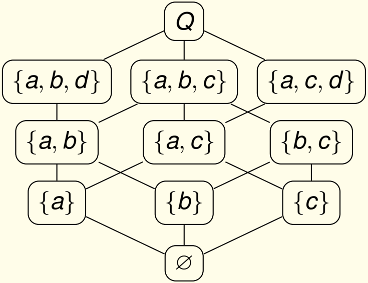
(ii)



Two examples of learning spaces on $Q = \{a, b, c, d\}$:

$\mathcal{L}^{(1)}$



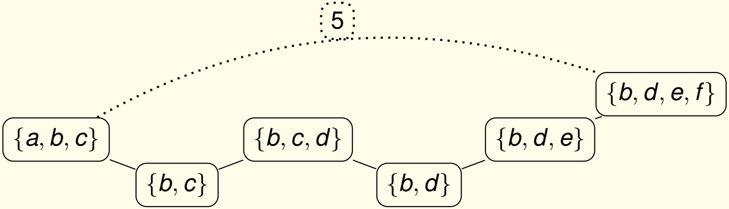


$\mathcal{L}^{(2)}$

There are many equivalent definitions of learning spaces,

that is many possible substitutes for Axioms (i) and (ii).

Learning Spaces, Knowledge Spaces (2/3)



Proposition (Cosyn and Uzun, 2009; Korte, Lovász and Schrader, 1991)

The pair $(Q, \mathcal{L})$ is a learning space if Q is a finite set, $\mathcal{L} \subseteq 2^Q$ and

(i') $\mathcal{L}$ is **stable under union** and contains $\varnothing$ and Q;

(ii') $\mathcal{L}$ is **well-graded**: $\forall K, L \in \mathcal{L},\ \exists K = L_0, L_1, \ldots, L_h = L$ with $L_i \in \mathcal{L},\ \ h = |K \triangle L|$ and $|L_{i-1} \triangle L_i| = 1$, for $1 \leq i \leq h$.

Definition

The pair $(Q, \mathcal{K})$ is a **knowledge space** if $\mathcal{K} \subseteq 2^Q$, $\varnothing, Q \in \mathcal{K}$ and $\mathcal{K}$ is stable under union.

Learning Spaces vs. Antimatroids (1/3)

For $\mathcal{K} \subseteq 2^Q$, define
$$\overline{\mathcal{K}} = \{ Q \setminus K \,|\, K \in \mathcal{K} \}.$$

Proposition

$(Q, \mathcal{K})$ is a knowledge space

$\iff$

$(Q, \overline{\mathcal{K}})$ is a 'closure space'.

Definition

A **closure space** $(Q, \mathcal{F})$ consists of
a **domain** Q, which is a finite set;
a collection $\mathcal{F}$ of subsets of Q, called the **closed sets**, s.t.:

(i) $\mathcal{F}$ is **closed under intersection** (F, $G \in \mathcal{F}$ implies $F \cap G \in \mathcal{F}$);

(ii) $\mathcal{F}$ contains both $\varnothing$ and Q.

Learning Spaces vs. Antimatroids (2/3)

Here is another way of looking at a closure space.

Definition

A **closure operator** on the set Q is a mapping $2^Q \to 2^Q : \; X \mapsto X^c$ satisfying

(i) $X \subseteq X^c$ (expansivity);

(ii) $X \subseteq Y$ implies $X^c \subseteq Y^c$ (monotonicity);

(iii) $(X^c)^c = X^c$ (idempotence);

(iv) $\varnothing^c = \varnothing$.

Any closure space $(Q, \mathcal{F})$ determines a closure operator with

$$X^c \; = \; \cap\{F \in \mathcal{F} \,|\, X \subseteq F\}.$$

Conversely, any closure operator $X \to X^c$ on Q determines a closure space $(Q, \mathcal{F})$ with

$$\mathcal{F} \; = \; \{F \in 2^Q \,|\, X^c = X\}.$$

Notice that the two constructions are mutual inverses.

Learning Spaces vs. Antimatroids (3/3)

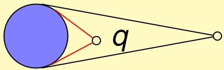
For $\mathcal{K} \subseteq 2^Q$, define
$$\overline{\mathcal{K}} = \{ Q \setminus K \mid K \in \mathcal{K} \}.$$

Proposition

$(Q, \mathcal{K})$ is a learning space $\iff$ $(Q, \overline{\mathcal{K}})$ is an 'antimatroid'.

Definition

An **antimatroid** $(E, \mathcal{C})$ is a closure space such that for all p, q in E and C in $\mathcal{C}$:
$$\big(q \in (\{p\} \cup C)^c \text{ and } q \neq p\big) \implies p \notin (\{q\} \cup C)^c\,.$$



See also Theorem 44 in Ganter and Wille (1996).

Antimatroids abound in math. and computer sci. (Korte, Lovász and Schrader, 1991).

Spaces from Prerequisite Relations

Assume R is a **prerequisite relation** on Q:

$p\,R\,q$ when the knowledge of p is necessary to acquire the knowledge of q.

Definition

A **state w.r.t.** R is any subset K of Q such that for all p, q in Q:

$$p\,R\,q \text{ and } q \in K \implies p \in K.$$

Proposition

Let $\mathcal{K}_R$ be the collection of all states w.r.t. R.

Then $(Q, \mathcal{K}_R)$ is always a knowledge space, and $\mathcal{K}$ is $\cap$-stable.

Moreover, $(Q, \mathcal{K}_R)$ is a learning space iff R is acyclic.

Remark

It makes sense to assume that R is a partial order on Q.

A Practical Problem

Suppose we know the collection Q of items in an area of knowledge; how do we build the collection of (potential) knowledge states?

The information comes

1. either from experts in the area;
2. or past assessment sessions of student knowledge.

We first treat the case of knowledge spaces,

then that of learning spaces (antimatroids).

Working always with the same domain Q, we say that

$\mathcal{K}$ is a knowledge space if $(Q, \mathcal{K})$ is a knowledge space,

$\mathcal{L}$ is a learning space if $(Q, \mathcal{L})$ is a learning space.

Building a Knowledge Space (1/3)

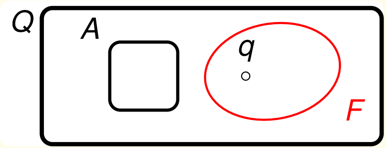
Typical query (A, q) to the expert, for some $A \subseteq Q$ and $q \in Q$:

Suppose that a student under examination has just provided wrong responses to all the items in A.

Is it practically certain that this student will also fail item q?

(Assume that the conditions are ideal in the sense that errors and lucky guesses are excluded.)

A positive response to query (A, q) rules out subsets from being potential knowledge states:



if the actual collection of states is $\mathcal{F}$, it rules out all of

$$\mathcal{D}_{\mathcal{F}}(A, q) \quad = \quad \{F \in \mathcal{F} \mid A \cap F = \varnothing \text{ and } q \in F\}.$$

Building a Knowledge Space (2/3)

The $\textsc{Query}$ **routine** asks sucessive queries (A, q) (without any repetition);

each time it collects a positive response, it rules out all of

$$\mathcal{D}_{\mathcal{F}}(A, q) \quad = \quad \{F \in \mathcal{F} \mid A \cap F = \varnothing \text{ and } q \in F\}.$$

If we limit ourselves to $|A| = 1$, we get the states of a (prerequisite) relation on Q;

if the expert is 'coherent', the relation is a partial order on Q.

In general, the $\textsc{Query}$ routine takes advantage of previous (positive and negative) responses to derive queries that do not need to be asked

(think of taking advantage of transitivity in the case of a relation).

A Theoretical Result (3/3)

Proposition (KOPPEN and DOIGNON, 1990)

Knowledge spaces $\mathcal{K}$ on Q,

and

relations $\mathcal{P}$ from $2^Q \setminus \{\varnothing\}$ to Q s.t. (for all q in Q and A, B in $2^Q \setminus \{\varnothing\}$):

(i) if $q \in A$, then $A \, \mathcal{P} \, q$;

(ii) if $A \, \mathcal{P} \, b$ for all b in B, and $B \, \mathcal{P} \, q$, then $A \, \mathcal{P} \, q$

are set in a one-to-one correspondence by

$$K \in \mathcal{K} \iff \big(\forall (A, q) \in \mathcal{P} : \; A \cap K = \varnothing \implies q \notin K\big),$$
$$(A, q) \in \mathcal{P} \iff \big(\forall K \in \mathcal{K} : \; A \cap K = \varnothing \implies q \notin K\big),$$

for all K, $A \in 2^Q$, $q \in Q$.

The correspondence derives from a Galois connection. Moreover, $\mathcal{P}$ “is” the closure operator of $\overline{\mathcal{K}}$.

For a similar problem, treated with much more mathematical sophistication, see GANTER (1999).

Building a Learning Space

Again, assume we have the whole set Q of items.

Problem

How to adapt the QUERY routine in order that it always produces a learning space?

I know of three solutions:

1.- in Eppstein Falmagne and Uzun (2009);

2.- in Falmagne and Doignon (2011), chapter 16 of *Learning Spaces*;

3.- one which was very recently conceived.

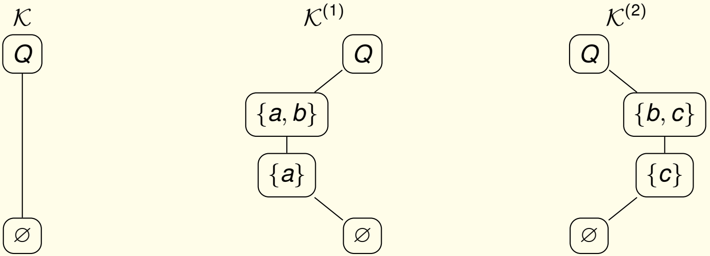
EPPSTEIN, FALMAGNE and UZUN (2009)

(Phase 1) Use the QUERY routine to build a knowledge space $\mathcal{K}$;

(Phase 2) add states until the enlarged $\mathcal{K}$ becomes a learning space.

Phase 1 eliminates subsets of Q;   Phase 2 adds subsets of Q.

In Phase 2, completion is (in general) not unique. With $Q = \{a, b, c\}$:



FALMAGNE and DOIGNON (2011) (1/4)

Basic idea: make sure that the actual collection of states remains a learning space.

▷ Start from a learning space (for instance 2^Q) and successively collect responses to queries;

▷ when a response is positive, do not at once delete states if the resulting space is not well-graded anymore.

There results the **adapted QUERY routine**.
(many details need explanations).

FALMAGNE and DOIGNON (2011) (2/4)

Let $\mathcal{K}$ be a knowledge space, and $K \in \mathcal{K}$.

Definition

The **inner fringe** of K is

$$K^{\mathcal{I}} = \{ q \in K \mid K \setminus \{q\} \in \mathcal{K} \}.$$

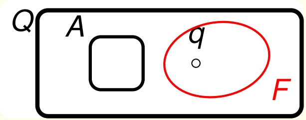
The **outer fringe** of K is

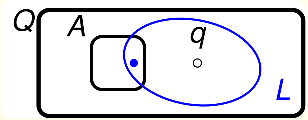
$$K^{\mathcal{O}} = \{ q \in Q \setminus K \mid K \cup \{q\} \in \mathcal{K} \}.$$

Notice the pedagogical interest of the outer fringe.

Falmagne and Doignon (2011) (3/4)

The following helps in the design of the adapted QUERY routine; remember that $\mathcal{D}_{\mathcal{K}}(A, q)$ is the collection of subsets ruled out by a positively answered query (A, q) (those F below on the left).





Proposition

For any knowledge space $\mathcal{K}$ and any query (A, q),
$\mathcal{K} \setminus \mathcal{D}_{\mathcal{K}}(A, q)$ is always a knowledge space.

If $\mathcal{L}$ is a learning space, then
$\mathcal{L} \setminus \mathcal{D}_{\mathcal{L}}(A, q)$ is a learning space
if and only if
there is no state L in $\mathcal{L}$ such that
$|L^{\mathcal{I}}| = 1, \quad A \cap L = L^{\mathcal{I}}, \quad$ and $\quad q \in L$.

FALMAGNE and DOIGNON (2011) (4/4)

Proposition

If $\mathcal{L}$ is a 'latent' learning space and the query responses are truthful with respect to $\mathcal{L}$,
then the adapted QUERY routine will ultimately uncover $\mathcal{L}$.

The proof is based on results of

EDELMAN and JAMISON (1985)

and

CASPARD and MONJARDET (2004)

about the collection of all antimatroids on a given set.

The Adjusted QUERY Routine (1/7)

Proposition (Caspard and Monjardet, 2004)

The collection of all learning spaces on Q, when ordered by inclusion, form a **sup-semilattice**.

Explanation: when $\mathcal{L}$ and $\mathcal{M}$ are two learning spaces (on Q), among all the learning spaces which contain $\mathcal{L}$ and $\mathcal{M}$ there is one, the **supremum** $\mathcal{L} \vee \mathcal{M}$, contained in all others.

Moreover

$$\mathcal{L} \vee \mathcal{M} = \{ L \cup M \mid L \in \mathcal{L},\, M \in \mathcal{M} \}$$

(not the union of $\mathcal{L}$ and $\mathcal{M}$).

By adding a minimum (new) element $\perp$, the collection of all learning spaces on Q becomes a lattice, with **infimum** given by

$$\mathcal{L} \wedge \mathcal{M} = \begin{cases} \vee\{ \mathcal{N} \mid \mathcal{N} \text{ is a learning space and } \mathcal{N} \subseteq \mathcal{L} \cap M \}, \\ \perp \qquad\qquad \text{if no such } \mathcal{N} \text{ exists.} \end{cases}$$

(For meditation)

The collection $\mathbb{L}$ of all learning spaces on Q is a well-graded collection of subsets of 2^Q, but not a learning space.

Ordered by inclusion, $\mathbb{L}$ forms a sup-semilattice.

With a new minimum element, $(\mathbb{L} \cup \{\bot\}, \subseteq)$ becomes a lattice.

The Adjusted QUERY Routine (2/7)

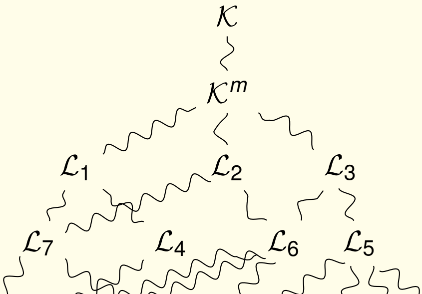
A byproduct of CASPARD and MONJARDET (2004):

Proposition

A knowledge space $\mathcal{K}$

either does not contain any learning space
(no subcollection of $\mathcal{K}$ forms a learning space),

or it contains a largest learning space $\mathcal{K}^m$
(among all learning spaces $\mathcal{L}$ with $\mathcal{L} \subseteq \mathcal{K}$,
there is $\mathcal{K}^m$ which contains all the other ones: $\mathcal{L} \subseteq \mathcal{K}^m$).



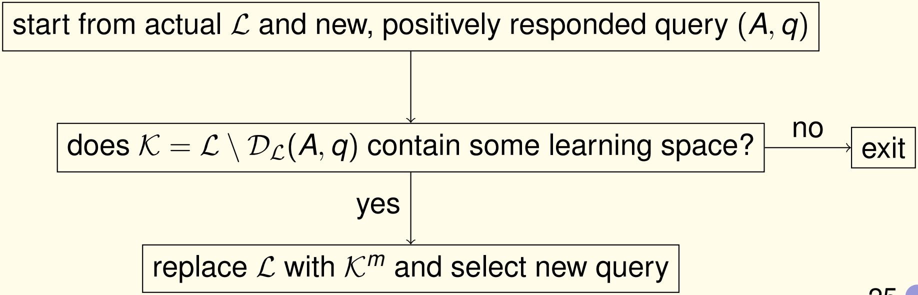
The Adjusted QUERY Routine (3/7)

▷ Start from a learning space (for instance 2^Q) and successively collect responses to queries;

▷ when the response to query (A, q) is positive and $\mathcal{L}$ is the actual learning space,
build the resulting knowledge space $\mathcal{K} = \mathcal{L} \setminus \mathcal{D}_{\mathcal{L}}(A, q)$ and check whether $\mathcal{K}$ contains some learning space:

if no, exit;

if yes, replace $\mathcal{L}$ with the largest learning space $\mathcal{K}^m$ contained in $\mathcal{K}$.



The Adjusted QUERY Routine (4/7)

In case of “exit”, there is no learning space satisfying all the responses to queries.

Proposition

If the expert has a ‘latent’ learning space $\mathcal{L}$ and
if his responses to queries are truthful w.r.t $\mathcal{L}$,

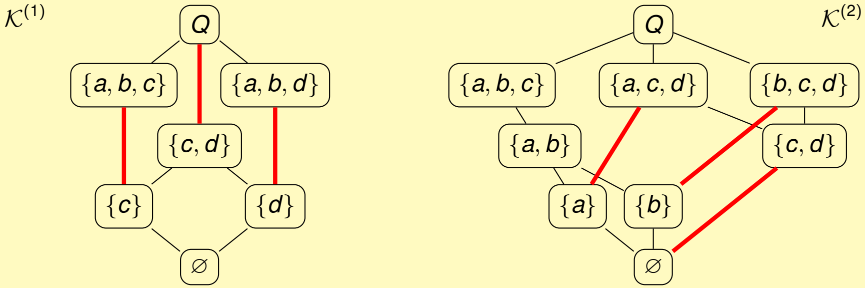
then the adjusted QUERY routine will ultimately uncover $\mathcal{L}$.

Given a knowledge space $\mathcal{K}$, how can we build the largest learning space $\mathcal{K}^m$ that $\mathcal{K}$ contains (if the latter exists)?

Let us look a two examples.

Example

Let $Q = \{a, b, c, d\}$, and $\mathcal{K}^{(1)}$, $\mathcal{K}^{(2)}$ be the two knowledge spaces



Here, $\mathcal{K}^{(1)}$ does not contain any learning space;

$\mathcal{K}^{(2)}$ contains three learning spaces, the largest one being

$$\mathcal{K}^m \quad = \quad \{\, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, Q \,\}.$$

Definition

In a knowledge space $\mathcal{K}$ (more precisely, $(Q, \mathcal{K})$),

a **learning path** is a chain of states (that is, a subcollection $\mathcal{C}$ of $\mathcal{K}$ such that any two states in $\mathcal{C}$ are comparable for inclusion);

a **gradation** is a learning path containing $|Q| + 1$ states.

Notice that any gradation is itself a learning space on Q.

Proposition

A knowledge space $\mathcal{K}$ contains some learning space iff it contains some gradation.

Then, the largest learning space $\mathcal{K}^m$ contained in $\mathcal{K}$ is the union of all gradations.

We leave aside many questions, in particular:

- how to work with the surmise function[1] (or the base), rather than the full learning space?
- how to design the most efficient algorithm?
- etc.

Thank you for your attention!

[1] As defined in my talk at ICFCA'13 in Dresde