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On close-to-convex functions satisfying a differential inequality

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Abstract. Let $\mathcal{C}_\alpha(\beta)$ denote the class of normalized functions f, analytic in the open unit disk $\mathbb{E}$ which satisfy the condition

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{(zf'(z))'}{\phi'(z)}\right]>\beta,\ z\in\mathbb{E},$$

where $\dfrac{f(z)f'(z)}{z}\neq 0,\ z\in\mathbb{E}$, ϕ is starlike and $\alpha,\ \beta$ are pre-assigned real numbers. In 1977, Chichra, P. N. [1] introduced and studied the class $\mathcal{C}_\alpha=\mathcal{C}_\alpha(0)$. He proved the members of class $\mathcal{C}_\alpha$ are close-to-convex for $\alpha\geq 0$. We here prove that functions in class $\mathcal{C}_\alpha(\beta)$ are close-to-convex for $-\dfrac{\alpha}{2}\Re\left(\dfrac{z\phi'(z)}{\phi(z)}\right)\leq\beta<1,\ \alpha\geq 0$ and the result is sharp in the sense that the constant β cannot be replaced by a real number smaller than $-\dfrac{\alpha}{2}\Re\left(\dfrac{\phi(z)}{z\phi'(z)}\right)$. We claim that our result improves the result of Chichra, P. N. [1].

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1. Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the classes of starlike and convex functions respectively analytically defined as follows:

$$\mathcal{S}^*=\left\{f\in\mathcal{A}:\Re\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}\right\},$$

and

$$\mathcal{K}=\left\{f\in\mathcal{A}:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in\mathbb{E}\right\}.$$

This is well-known that

$$f(z)\in\mathcal{K}\Leftrightarrow zf'(z)\in\mathcal{S}^*. \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be close to convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$ and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0,\ z \in \mathbb{E}.$$

In view of the relation (1.1), the above definition takes the following form in case g is normalized. A function $f \in \mathcal{A}$ is said to be close to convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$, and a starlike function ϕ such that

$$\Re\left(e^{i\alpha}\frac{zf'(z)}{\phi(z)}\right) > 0,\ z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [3] and Warchawski [4] obtained a simple but elegant criterion for univalence of analytic functions. They proved that if an analytic function f satisfies $\Re f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.
Let $\mathcal{C}_\alpha(\beta)$ denote the class of normalized analytic functions f which satisfy the condition

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] > \beta,\ z \in \mathbb{E},$$

where $\dfrac{f(z)f'(z)}{z} \neq 0,\ z \in \mathbb{E}$, ϕ is starlike and $\alpha,\ \beta$ are pre-assigned real numbers. The class $\mathcal{C}_\alpha = \mathcal{C}_\alpha(0)$ was introduced and studied by Chichra, P. N. [1] in 1977. He called the members of class $\mathcal{C}_\alpha$ as $\alpha-$ close-to-convex functions. Infact, he proved the following result.

Theorem 1.1. *Let $f(z) \in \mathcal{C}_\alpha$ and $\alpha \geq 0$. Then $f(z)$ is close-to-convex in $\mathbb{E}$.*

In the present paper, we establish the result that functions in $\mathcal{C}_\alpha(\beta)$ are close-to-convex for $-\dfrac{\alpha}{2}\Re\left(\dfrac{z\phi'(z)}{\phi(z)}\right) \leq \beta < 1,\ \alpha \geq 0$. Our result is the best possible in the sense that the constant β cannot be replaced by a real number smaller than $-\dfrac{\alpha}{2}\Re\left(\dfrac{\phi(z)}{z\phi'(z)}\right)$.
We also claim that our result improves the result of Chichra, P. N. [1]. To prove our main result, we shall use the following lemma of Miller [2].

Lemma 1.2. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ϕ satisfy the following conditions:*
(i) $\phi(u, v)$ is continuous in $\mathbb{D}$
(ii) $(1, 0) \in \mathbb{D}$ and $\Re[\phi(1, 0)] > 0$ and
(iii) $\Re[\phi(iu_2, v_1)] \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1 + u_2^2)/2$.
Let $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be regular in the open unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\Re[\phi(p(z), zp'(z))] > 0,\ z \in \mathbb{E},$$

then $\Re p(z) > 0,\ z \in \mathbb{E}$.

2. Main result

Theorem 2.1. *Let α and β be real numbers such that $\alpha \geq 0$ and*

$$-\frac{\alpha}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right) \leq \beta < 1$$

for a starlike function ϕ. Assume that $f \in \mathcal{A}$ satisfies

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] > \beta, \ z \in \mathbb{E}, \tag{2.1}$$

then $\Re\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ in $\mathbb{E}$ and hence f is close-to-convex and hence univalent in $\mathbb{E}$. The result is sharp in the sense that the constant β on the right hand side of (2.1) cannot be replaced by a real number smaller than $-\frac{\alpha}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right)$.

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in $\mathbb{E}$ such that for all $z \in \mathbb{E}$, we write

$$\frac{zf'(z)}{\phi(z)} = p(z).$$

Then,

$$(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)} = p(z) + \alpha z p'(z)\frac{\phi(z)}{z\phi(z)}.$$

Therefore, condition (2.1) is equivalent to

$$\Re\left(\frac{1}{1-\beta}p(z) + \frac{\alpha}{1-\beta}zp'(z)\frac{\phi(z)}{z\phi'(z)} - \frac{\beta}{1-\beta}\right) > 0, \ z \in \mathbb{E}. \tag{2.2}$$

For $\mathbb{D} = \mathbb{C} \times \mathbb{C}$, define $\Phi(u,v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\Phi(u,v) = \frac{1}{1-\beta}u + \frac{\alpha}{1-\beta}v\frac{\phi(z)}{z\phi'(z)} - \frac{\beta}{1-\beta}, \ z \in \mathbb{E}.$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0) \in \mathbb{D}$ and $\Re\left(\Phi(1,0)\right) = 1 > 0$. Further, in view of (2.2), we get, $\Re[\Phi(p(z), zp'(z))] > 0, \ z \in \mathbb{E}$. Let $u = u_1 + iu_2, v = v_1 + iv_2$ where u_1, u_2, v_1 and v_2 are all real numbers. Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\frac{1+u_2^2}{2}$, we have

$$\begin{aligned}
\Re\Phi(iu_2, v_1) &= \Re\left(\frac{1}{1-\beta}u_2 i + \frac{\alpha}{1-\beta}v_1\frac{\phi(z)}{z\phi'(z)} - \frac{\beta}{1-\beta}\right) \\
&\leq -\left[\frac{\alpha}{1-\beta}\frac{1+u_2^2}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right) + \frac{\beta}{1-\beta}\right] \\
&\leq -\left[\frac{\alpha}{2(1-\beta)}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right) + \frac{\beta}{1-\beta}\right] \\
&\leq 0.
\end{aligned}$$

In view of (2.2) and Lemma 1.2, proof now follows.

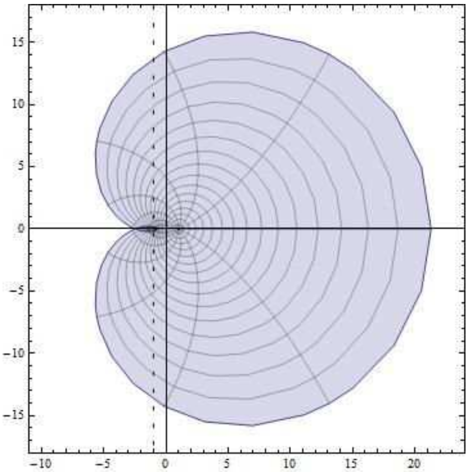

Figure 2.1

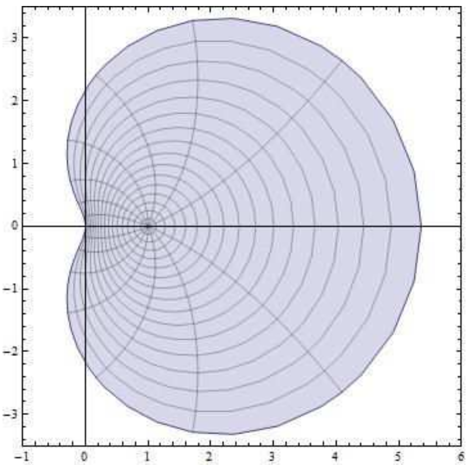

Figure 2.2

To show that the constant β on the right hand side of (2.1) cannot be replaced by a real number smaller than $-\dfrac{\alpha}{2}\Re\left(\dfrac{\phi(z)}{z\phi'(z)}\right)$, we consider the function $f(z) = z\, e^z \in \mathcal{A}$ and $\phi(z) = z \in \mathcal{S}^*$. Using Mathematica 9.0, we plot, in Figure 2.1, the image of the unit disk under the operator $(1-\alpha)\dfrac{zf'(z)}{\phi(z)} + \alpha\dfrac{(zf'(z))'}{\phi'(z)}$ taking $\alpha = 2$. From this figure, we notice that minimum real part of $(1-\alpha)\dfrac{zf'(z)}{\phi(z)} + \alpha\dfrac{(zf'(z))'}{\phi'(z)}$ is smaller than -1 (the calculated value of $-\frac{\alpha}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right)$ for $\alpha = 2$ and $\phi(z) = z$). In Figure 2.2, we plot the image of unit disk under the operator $\frac{zf'(z)}{\phi(z)}$. It is obvious that $\Re\left(\frac{zf'(z)}{\phi(z)}\right) \not> 0$ for all z in $\mathbb{E}$. For example, the point $z = -\frac{1}{2} + i\frac{\pi}{4}$ is an interior point of $\mathbb{E}$, but at this point $\Re\left(\frac{zf'(z)}{\phi(z)}\right) = -\frac{\pi - 2}{4\sqrt{2e}} = -0.1224\cdots < 0$. This justifies our claim. □

Remark 2.2. We claim that our result improves the result of Chichra, P. N. [1]. In fact, when we take $f(z) = -z - 2\log(1-z) \in \mathcal{A}$, $\phi(z) = z$ and $\alpha = 2$ in Theorem 2.1, we notice that at $z = -1$,

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] = -1.$$

Thus the function f does not satisfy the hypothesis of Theorem 1.1 due to Chichra, P. N. [1] i.e. $f \notin \mathcal{C}_\alpha$ although $\Re\left(\frac{zf'(z)}{\phi(z)}\right) = \Re\left(\frac{1+z}{1-z}\right) > 0$ in $\mathbb{E}$. Hence the result of Chichra, P. N. [1] fails to conclude the close-to-convexity in this case whereas Theorem 2.1 concludes the same.

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