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Some Hermite-Hadamard type inequalities for functions whose exponentials are convex

Silvestru Sever Dragomir and Ian Gomm

Abstract. Some inequalities of Hermite-Hadamard type for functions whose exponentials are convex are obtained.

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1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [1] and the references therein.

We denote by $\mathfrak{Expconv}(I)$ the class of all functions defined on the interval I of real numbers such that $\exp(f)$ is convex on I. If $\mathfrak{Conv}(I)$ is the class of convex functions defined on I then we have the following fact:

Proposition 1.1. *We have the strict inclusion*

$$\mathfrak{Conv}(I) \subsetneq \mathfrak{Expconv}(I).$$

Proof. If f is convex , then $\exp(f)$ is log-convex and therefore convex on I and the inclusion is proved.

For $r \geq 1$ the function $f_r(x) = r\ln x,\ x>0$ is concave on $(0,\infty)$. We have $\exp(f_r(x)) = x^r$ is a convex function, therefore $f_r \in \mathfrak{Expconv}(I)\setminus\mathfrak{Conv}(I)$. □

We observe that for twice differentiable functions g on $\mathring{I}$, the interior of I we have that

$$(\exp(g(x)))'' = \left([g'(x)]^2 + g''(x)\right) \exp g(x), \ x \in \mathring{I},$$

therefore $g \in \mathfrak{Expconv}(I)$ if and only if

$$[g'(x)]^2 + g''(x) \geq 0 \text{ for any } x \in \mathring{I}.$$

2. Some Hermite-Hadamard type inequalities

Now, if $g \in \mathfrak{Expconv}(I)$, then by the Hermite-Hadamard inequality for $\exp(g)$ we have for $a, b \in I$ with $a < b$ that

$$\exp g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \exp g(t)\,dt \leq \frac{1}{2}[\exp g(a) + \exp g(b)]. \tag{2.1}$$

By Jensen's integral inequality for the exp function we also have for any integrable function $h : [a, b] \to \mathbb{R}$ that

$$\exp\left(\frac{1}{b-a}\int_a^b h(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \exp h(t)\,dt. \tag{2.2}$$

We define the logarithmic mean as

$$L = L(a, b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b, \end{cases} \qquad a, b > 0.$$

We can improve the inequality (2.1) for convex functions as follows:

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then we have for $f(b) \neq f(a)$ the inequalities*

$$\begin{aligned} \exp f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \exp f(t)\,dt \\ &\leq \frac{\exp f(b) - \exp f(a)}{f(b) - f(a)}\left(\leq \frac{1}{2}[\exp f(a) + \exp f(b)]\right). \end{aligned} \tag{2.3}$$

Proof. The first inequality follows by Hermite-Hadamard inequality for the convex function f. The second inequality follows by (2.2).

It is know that if g is log convex, then by [2]

$$\frac{1}{b-a}\int_a^b g(t)\,dt \leq L(g(a), g(b)). \tag{2.4}$$

Since f is convex, then $g = \exp(f)$ is log-convex and by (2.4) we get the third inequality in (2.3). $\square$

A recent paper connected with such results is [4].

Consider the *identric mean* of two positive numbers

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \qquad a, b > 0.$$

We observe that

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln u du$$

for $a, b > 0$, $a \neq b$.

The following result holds:

Theorem 2.2. *Assume that* $f \in \mathfrak{Expconv}(I)$ *and* $a, b \in I$ *with* $a < b$. *Then we have*

$$\exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq I(\exp f(a), \exp f(b)) \tag{2.5}$$

and

$$\begin{aligned} &\exp f\left(\frac{a+b}{2}\right) \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x) + \exp f(a+b-x)}{2}\right] dx\right) \\ &\leq \frac{1}{b-a}\int_a^b \exp f(x)\,dx. \end{aligned} \tag{2.6}$$

Proof. Since $f \in \mathfrak{Expconv}(I)$, then

$$\exp f((1-\lambda)a + \lambda b) \leq (1-\lambda)\exp f(a) + \lambda \exp f(b)$$

for any $\lambda \in [0,1]$, which is equivalent to

$$f((1-\lambda)a + \lambda b) \leq \ln[(1-\lambda)\exp f(a) + \lambda \exp f(b)] \tag{2.7}$$

for any $\lambda \in [0,1]$.

Integrating (2.7) on $[0,1]$ we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &= \int_0^1 f((1-\lambda)a + \lambda b)\,d\lambda \\ &\leq \int_0^1 \ln[(1-\lambda)\exp f(a) + \lambda \exp f(b)]\,d\lambda \\ &= \frac{1}{\exp f(b) - \exp f(a)}\int_{\exp f(a)}^{\exp f(b)} \ln u du \\ &= \ln I(\exp f(a), \exp f(b)) \end{aligned} \tag{2.8}$$

and the inequality in (2.5) is proved.

From (2.7) we have

$$f\left(\frac{x+y}{2}\right) \leq \ln\left[\frac{\exp f(x)+\exp f(y)}{2}\right] \tag{2.9}$$

for any $x, y \in I$.

From (2.9) we have

$$f\left(\frac{a+b}{2}\right) \leq \ln\left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right] \tag{2.10}$$

for any $x \in [a,b]$.

Integrating the inequality (2.10) over x on $[a,b]$ we get the first inequality in (2.6).

By the Jensen's inequality for the concave function ln we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right]dx \qquad (2.11)\\
&\leq \ln\left(\frac{1}{b-a}\int_a^b \left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right]dx\right)\\
&= \ln\left(\frac{1}{2(b-a)}\int_a^b [\exp f(x)+\exp f(a+b-x)]\,dx\right)\\
&= \ln\left(\frac{1}{b-a}\int_a^b \exp f(x)\,dx\right)
\end{aligned}$$

and the second inequality in (2.6) is proved. □

If we consider *Toader's mean* defined as (see for instance [5] and [7] for many relations of this mean with other means)

$$E = E(a,b) := \begin{cases} a & \text{if} \quad a=b, \\ \log I(\exp a, \exp b) & \text{if} \quad a \neq b, \end{cases} \qquad a,b \in \mathbb{R}.$$

we can write (2.5) in an equivalent form as

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq E(\exp f(a), \exp f(b)). \tag{2.12}$$

Remark 2.3. If the function $g : I \to (0,\infty)$ is convex on I, then $f = \ln g \in \mathfrak{Expconv}(I)$ and for $a, b \in I$ with $a < b$ we have, by (2.5) and (2.6), the following inequalities

$$\exp\left(\frac{1}{b-a}\int_a^b \ln g(t)\,dt\right) \leq I(g(a), g(b)) \tag{2.13}$$

and

$$g\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{g(x)+g(a+b-x)}{2}\right]dx\right) \tag{2.14}$$
$$\leq \frac{1}{b-a}\int_a^b g(x)\,dx.$$

3. Related results

The following related result also holds:

Theorem 3.1. *Assume that $f \in \mathfrak{Expconv}(I)$ and $a, b \in I$ with $a < b$. Then we have*

$$\frac{f(a)(x-a)+f(b)(b-x)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \tag{3.1}$$
$$\geq \exp f(x)\left[\exp(-f(x)) - \frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right]$$

for any $x \in [a, b]$.

In particular, we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy \tag{3.2}$$
$$\geq \exp f\left(\frac{a+b}{2}\right)\left[\exp\left(-f\left(\frac{a+b}{2}\right)\right) - \frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right].$$

Proof. Since the function $\exp(f)$ is convex, it has lateral derivatives in each point of (a, b) and $f = \ln(\exp f)$ does the same. Then for any $x, y \in (a, b)$ we have

$$\exp f(x) - \exp f(y) \geq f'_-(y)(x-y)\exp f(y)$$

and dividing by $\exp f(y) > 0$ we get

$$\exp f(x)\exp[-f(y)] - 1 \geq f'_-(y)(x-y) \tag{3.3}$$

for any $x, y \in (a, b)$.

Integrating (3.3) over y on $[a, b]$ and dividing by $b-a$ we get

$$\exp f(x)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy - 1 \tag{3.4}$$
$$\geq \frac{1}{b-a}\int_a^b f'_-(y)(x-y)\,dy$$
$$= \frac{1}{b-a}\left[f(y)(x-y)\big|_a^b + \int_a^b f(y)\,dy\right]$$
$$= \frac{1}{b-a}\left[\int_a^b f(y)\,dy - f(a)(x-a) - f(b)(b-x)\right]$$

for any $x \in [a, b]$, which is equivalent to the desired inequality (3.1). □

Corollary 3.2. *With the assumptions of Theorem* 3.1 *we have*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy \qquad (3.5)\\
&\geq \frac{\exp f(a)+\exp f(b)}{2}\left[1-\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right].
\end{aligned}$$

Proof. If we take $x=a$ and $x=b$ in (3.4) we get

$$\exp f(a)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\geq \frac{1}{b-a}\int_a^b f(y)\,dy-f(b)$$

and

$$\exp f(b)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\geq \frac{1}{b-a}\int_a^b f(y)\,dy-f(a).$$

Adding these inequalities and dividing by two we get

$$\begin{aligned}
&\frac{\exp f(a)+\exp f(b)}{2}\left[\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\right]\\
&\geq \frac{1}{b-a}\int_a^b f(y)\,dy-\frac{f(a)+f(b)}{2},
\end{aligned}$$

which is equivalent to the desired inequality (3.5). □

Corollary 3.3. *With the assumptions of Theorem* 3.1 *and if*

$$x_0:=\frac{f(b)\,b-f(a)\,a-\int_a^b f(y)\,dy}{f(b)-f(a)}\in[a,b], \qquad (3.6)$$

where $f(b)\neq f(a)$, *then we have*

$$\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\geq \exp\left(-f\left(\frac{f(b)\,b-f(a)\,a-\int_a^b f(y)\,dy}{f(b)-f(a)}\right)\right). \qquad (3.7)$$

Proof. Follows by (3.1) by taking $x=x_0$ defined in (3.6). □

The inequality (3.7) can be found in Sándor's paper [3] where x_0 considered in (3.6) is in fact a mean called by him as "*generated by derivatives of functions*". This mean is extended in [9] (see also [6]), and generalized many results related to integral inequalities. See also [8] for more results.

Remark 3.4. Since

$$x_0=\frac{\int_a^b f'(y)\,y\,dy}{\int_a^b f'(y)\,dy},$$

then a sufficient condition for (3.6) to hold is that f is monotonic nondecreasing or nonincreasing on the whole interval $[a,b]$.

Remark 3.5. If the function $g : I \to (0, \infty)$ is convex on I, then $f = \ln g \in \mathfrak{Expconv}\,(I)$ and for $a, b \in I$ with $a < b$ we have, by (3.1), (3.2) and (3.5), the following inequalities

$$\ln\left([g(a)]^{\frac{x-a}{b-a}}\,[g(b)]^{\frac{b-x}{b-a}}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \tag{3.8}$$
$$\geq g(x)\left[\frac{1}{g(x)} - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right],$$

$$\ln\left(\sqrt{g(a)\,g(b)}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \tag{3.9}$$
$$\geq g\left(\frac{a+b}{2}\right)\left[\frac{1}{g\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right],$$

and

$$\ln\left(\sqrt{g(a)\,g(b)}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \tag{3.10}$$
$$\geq \frac{g(a)+g(b)}{2}\left[1 - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right].$$

If

$$x_0 := \frac{\ln\left(\frac{[g(b)]^b}{[g(a)]^a}\right) - \int_a^b \ln g(y)\,dy}{\ln\left(\frac{g(b)}{g(a)}\right)} \in [a, b], \tag{3.11}$$

where $g(b) \neq g(a)$, then we have

$$\frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy \geq \frac{1}{g\left(\frac{\ln\left(\frac{[g(b)]^b}{[g(a)]^a}\right) - \int_a^b \ln g(y)dy}{\ln\left(\frac{g(b)}{g(a)}\right)}\right)}. \tag{3.12}$$

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Silvestru Sever Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia

School of Computational & Applied Mathematics
University of the Witwatersrand, Private Bag 3
Johannesburg 2050, South Africa
e-mail: `sever.dragomir@vu.edu.au`

Ian Gomm
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
e-mail: `ian.gomm@vu.edu.au`