Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 403–411

Pascu-type p-valent functions associated with the convolution structure

Birgül Öner and Sevtap Sümer Eker

Abstract. Making use of convolution structure, we introduce a new class of p-valent functions. Among the results presented in this paper include the coefficient bounds, distortion inequalities, extreme points and integral means inequalities for this generalized class of functions are discussed.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: p-valent functions, coefficient bounds, Hadamard product (or convolution), extreme points, distortion bounds, integral means, Sălăgean operator.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=2p+1}^{\infty} a_k z^k. \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

which are *analytic* and *p-valent* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

A function $f \in \mathcal{A}_p$ is β-Pascu convex of order α if

$$\frac{1}{p} Re \left\{ \frac{(1-\beta) z f'(z) + \frac{\beta}{p} z \left(z f'(z) \right)'}{(1-\beta) f(z) + \frac{\beta}{p} z f'(z)} \right\} > \alpha \qquad (0 \le \beta \le 1,\ 0 \le \alpha < 1).$$

In the other words $(1-\beta) f(z) + \frac{\beta}{p} z f'(z)$ is in $f \in \mathcal{S}_p^*$ the class of p-valent starlike functions (for details [5], see also [1], [3]).

Given two functions $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is given by

$$g(z) = z^p + \sum_{k=2p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}),$$

the *Hadamard product* (or *convolution*) $f * g$ is defined (as usual) by

$$(f * g)(z) = z^p + \sum_{k=2p+1}^{\infty} a_k b_k z^k = (g * f)(z) \ , \ z \in \mathbb{U}. \tag{1.2}$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

See also Duren [2].

On the other hand, Sălăgean [6] introduced the following operator which is popularly known as the *Sălăgean derivative operator*:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

and, in general,

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

We easily find from (1.1) that

$$D^n f(z) = p^n z^p + \sum_{k=2p+1}^{\infty} k^n a_k z^k \qquad (f \in \mathcal{A}_p \ ; \ n \in \mathbb{N}_0).$$

We denote by $\mathcal{T}_p$ the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=2p+1}^{\infty} a_k z^k, \quad (a_k \geq 0, \ p \in \mathbb{N}) \tag{1.3}$$

which are p-valent in $\mathbb{U}$.

For a given function $g \in \mathcal{A}_p$ defined by

$$g(z) = z^p + \sum_{k=2p+1}^{\infty} b_k z^k \quad (b_k > 0, \ p \in \mathbb{N}), \tag{1.4}$$

we introduce here a new class $\mathcal{AS}_g^*(n, p, \alpha, \beta)$ of functions belonging to the subclass of $\mathcal{T}_p$ which consists of functions $f(z)$ of the form (1.3) satisfying the following inequality:

$$\frac{1}{p} Re \left\{ \frac{(1-\beta)D^{n+1}(f*g)(z) + \frac{\beta}{p} D^{n+2}(f*g)(z)}{(1-\beta)D^n(f*g)(z) + \frac{\beta}{p} D^{n+1}(f*g)(z)} \right\} > \alpha \tag{1.5}$$

$$(0 \leq \alpha < 1, \ 0 \leq \beta \leq 1, \ n, p \in \mathbb{N})$$

In this paper, we obtain the coefficient inequalities, distortion theorems as well as integral means inequalities for functions in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$.

We first prove a necessary and sufficient condition for functions to be in $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ in the following:

2. Coefficient inequalities

Theorem 2.1. *A function $f(z)$ given by (1.3) is in $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ if and only if for $0 \le \alpha < 1$, $0 \le \beta \le 1$, $n,p \in \mathbb{N}$,*

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\, k^n a_k b_k \le p^{n+2}(1-\alpha). \tag{2.1}$$

Proof. Assume that $f \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then, in view of (1.3) to (1.5), we have

$$\frac{1}{p} Re \left\{ \frac{(1-\beta)D^{n+1}(f*g)(z) + \frac{\beta}{p}D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z) + \frac{\beta}{p}D^{n+1}(f*g)(z)} \right\}$$

$$= \frac{1}{p} Re \left\{ \frac{p^{n+1} - \sum\limits_{k=2p+1}^{\infty} \left[(1-\beta+\frac{\beta}{p}k)\right] k^{n+1} a_k b_k z^{k-p}}{p^{n} - \sum\limits_{k=2p+1}^{\infty} \left[(1-\beta+\frac{\beta}{p}k)\right] k^{n} a_k b_k z^{k-p}} \right\} > \alpha \qquad (z \in \mathbb{U}).$$

If we choose z to be real and let $r \to 1^-$, the last inequality leads us to desired assertion (2.1) of Theorem 2.1.

Conversely, assume that (2.1) holds. For $f(z) \in \mathcal{A}_p$, let us define the function $F(z)$ by

$$F(z) = \frac{1}{p} \frac{(1-\beta)D^{n+1}(f*g)(z) + \frac{\beta}{p}D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z) + \frac{\beta}{p}D^{n+1}(f*g)(z)} - \alpha$$

It suffices to show that

$$\left| \frac{F(z)-1}{F(z)+1} \right| < 1 \qquad (z \in \mathbb{U}).$$

We note that

$$\left| \frac{F(z)-1}{F(z)+1} \right|$$

$$= \left| \frac{(1-\beta)D^{n+1}(f*g)(z) + \frac{\beta}{p}D^{n+2}(f*g)(z) - p(\alpha+1)\left[(1-\beta)D^{n}(f*g)(z) + \frac{\beta}{p}D^{n+1}(f*g)(z)\right]}{(1-\beta)D^{n+1}(f*g)(z) + \frac{\beta}{p}D^{n+2}(f*g)(z) - p(\alpha-1)\left[(1-\beta)D^{n}(f*g)(z) + \frac{\beta}{p}D^{n+1}(f*g)(z)\right]} \right|$$

$$= \left| \frac{-\alpha p^{n+1} - \sum\limits_{k=2p+1}^{\infty} \left[(k-\alpha p-p)(1-\beta+\frac{\beta}{p}k)\right] k^{n} a_k b_k z^{k-p}}{(2-\alpha)p^{n+1} - \sum\limits_{k=2p+1}^{\infty} \left[(k-\alpha p+p)(1-\beta+\frac{\beta}{p}k)\right] k^{n} a_k b_k z^{k-p}} \right|$$

$$\leq \frac{\alpha p^{n+2} + \sum_{k=2p+1}^{\infty} \left[(k - \alpha p - p)(p - \beta p + \beta k)\right] k^n a_k b_k}{(2-\alpha)p^{n+2} - \sum_{k=2p+1}^{\infty} \left[(k - \alpha p + p)(p - \beta p + \beta k)\right] k^n a_k b_k}$$

The last expression is bounded above by 1, if

$$\alpha p^{n+2} + \sum_{k=2p+1}^{\infty} \left[(k - \alpha p - p)(p - \beta p + \beta k)\right] k^n a_k b_k$$

$$\leq (2-\alpha)p^{n+2} - \sum_{k=2p+1}^{\infty} \left[(k - \alpha p + p)(p - \beta p + \beta k)\right] k^n a_k b_k$$

which is equivalent to our condition (2.1). This completes the proof of our theorem. □

Corollary 2.2. *Let $f(z)$ given by (1.3). If $f \in \mathcal{AS}_g^*(n, p, \alpha, \beta)$, then*

$$a_k \leq \frac{p^{n+2}(1-\alpha)}{\left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n b_k} \tag{2.2}$$

with equality for functions of the form

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{\left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n b_k} z^k$$

Proof. If $f \in \mathcal{AS}_g^*(n, p, \alpha, \beta)$, then by making use of (2.1), we obtain

$$\begin{aligned} \left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n a_k b_k &\leq \sum_{k=2p+1}^{\infty} \left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n a_k b_k \\ &\leq p^{n+2}(1-\alpha) \end{aligned}$$

or

$$a_k \leq \frac{p^{n+2}(1-\alpha)}{\left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n b_k}.$$

Clearly for

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{\left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n b_k} z^k,$$

we have

$$a_k = \frac{p^{n+2}(1-\alpha)}{\left[(k - \alpha p)(p - \beta p + \beta k)\right] k^n b_k}.$$

□

3. Distortion inequalities

In this section, we shall prove distortion theorems for functions belonging to the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$.

Theorem 3.1. *Let the function $f(z)$ of the form (1.3) be in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$|f(z)| \geq r^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1} \tag{3.1}$$

and

$$|f(z)| \leq r^p + \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1}, \tag{3.2}$$

provided $b_k \geq b_{2p+1}$ $(k \geq 2p+1)$. The result is sharp with equality for

$$f(z) = z^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} z^{2p+1}.$$

at $z=r$ and $z=re^{\frac{i(2m+1)\pi}{p+1}}$, $m \in \mathbb{Z}$.

Proof. Since $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$, we apply Theorem 2.1, we obtain

$$(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1} \sum_{k=2p+1}^{\infty} a_k$$

$$\leq \sum_{k=2p+1}^{\infty} \left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n a_k b_k \leq p^{n+2}(1-\alpha).$$

Thus, we obtain

$$\sum_{k=2p+1}^{\infty} a_k \leq \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}}. \tag{3.3}$$

From (1.3) and (3.3), we have

$$|f(z)| \leq |z|^p + |z|^{2p+1} \sum_{k=2p+1}^{\infty} a_k \leq r^p + \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1}$$

and

$$|f(z)| \geq |z|^p - |z|^{2p+1} \sum_{k=2p+1}^{\infty} a_k \geq r^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1}.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the function $f(z)$ of the form (1.3) be in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$|f'(z)| \geq pr^{p-1} - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1} b_{2p+1}} r^{2p} \tag{3.4}$$

and

$$|f'(z)| \leq pr^p + \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1}b_{2p+1}}r^{2p}, \tag{3.5}$$

provided $b_k \geq b_{2p+1}$ $(k \geq 2p+1)$. *The result is sharp with equality for*

$$f(z) = z^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1}b_{2p+1}}z^{2p}$$

at $z = r$ *and* $z = re^{\frac{i(2m+1)\pi}{p}}$, $m \in \mathbb{Z}$.

Proof. From Theorem 2.1 and (3.3), we have

$$\sum_{k=2p+1}^{\infty} ka_k \leq \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1}b_{2p+1}}.$$

and the remaining part of the proof is similar to the proof of Theorem 3.1. □

4. Extreme points

Theorem 4.1. *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}z^k$$

$$(b_k > 0, 0 \leq \alpha < 1, 0 \leq \beta \leq 1,\ n, p \in \mathbb{N},\ k = 2p+1, 2p+2, ...)\,.$$

Then $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$ *if and only if it can be expressed in the following form:*

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_p \geq 0$, $\lambda_k \geq 0$ *and* $\lambda_p + \sum_{k=2p+1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z) = z^p - \sum_{k=2p+1}^{\infty} \lambda_k \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}z^k.$$

Then from Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n \lambda_k \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}b_k$$

$$= \sum_{k=2p+1}^{\infty} \lambda_k p^{n+2}(1-\alpha) = p^{n+2}(1-\alpha)(1-\lambda_p) \leq p^{n+2}(1-\alpha)$$

Thus, in view of Theorem 2.1, we find that $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$.
Conversely, suppose that $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then, since

$$a_k \leq \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k} \qquad (p \in \mathbb{N}),$$

we may set

$$\lambda_k = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}{p^{n+2}(1-\alpha)} a_k \quad (p\in\mathbb{N})$$

and

$$\lambda_p = 1 - \sum_{k=2p+1}^{\infty} \lambda_k.$$

Thus, clearly, we have

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z).$$

This completes the proof of theorem. □

Corollary 4.2. *The extreme points of the class* $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ *are given by*

$$f_p(z) = z^p$$

and

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k} z^k, \qquad (k\geq 2p+1,\ p\in\mathbb{N})\,. \tag{4.1}$$

Theorem 4.3. *The class* $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ *is a convex set.*

Proof. Suppose that each of the functions $f_i(z),\ (i=1,2)$ given by

$$f_i(z) = z^p - \sum_{k=2p+1}^{\infty} a_{k,i} z^k, \qquad (a_{k,i}\geq 0)$$

is in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \eta f_1(z) + (1-\eta) f_2(z), \qquad (0\leq \eta < 1)$$

is also in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. Since

$$g(z) = \eta\left(z^p - \sum_{k=2p+1}^{\infty} a_{k,1} z^k\right) + (1-\eta)\left(z^p - \sum_{k=2p+1}^{\infty} a_{k,2} z^k\right)$$

$$= z^p - \sum_{k=2p+1}^{\infty} [\eta a_{k,1} + (1-\eta) a_{k,2}]\, z^k$$

with the aid of Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n\,[\eta a_{k,1}+(1-\eta)a_{k,2}]\,b_k$$

$$= \eta \sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n a_{k,1} b_k$$

$$+(1-\eta) \sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n a_{k,2} b_k$$

$$\leq \eta p^{n+2}(1-\alpha) + (1-\eta)p^{n+2}(1-\alpha) = p^{n+2}(1-\alpha). \qquad \square$$

5. Integral means inequalities

In 1925, Littlewood proved the following subordination theorem.

Theorem 5.1. (Littlewood [4]) *If f and g are analytic in $\mathbb{U}$ with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

We will make use of Theorem 5.1 to prove

Theorem 5.2. *Let $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$ and $f_k(z)$ is defined by (4.1). If there exists an analytic function $w(z)$ given by*

$$w(z)^{k-p} = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p},$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|f_k(re^{i\theta})\right|^{\mu} d\theta \qquad (\mu > 0).$$

Proof. We must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=2p+1}^{\infty} a_k z^{k-p}\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|1 - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k} z^{k-p}\right|^{\mu} d\theta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1 - \sum_{k=2p+1}^{\infty} a_k z^{k-p} \prec 1 - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k} z^{k-p}.$$

By setting

$$1 - \sum_{k=2p+1}^{\infty} a_k z^{k-p} = 1 - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k} [w(z)]^{k-p},$$

we find that

$$[w(z)]^{k-p} = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p}$$

which readily yields $w(0) = 0$.

Furthermore, using (2.1), we obtain

$$|w(z)|^{k-p} \leq \left|\frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p}\right|$$

$$\leq \frac{\left[(k-\alpha p)(p-\beta p+\beta k)\right]k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k \left|z\right|^{k-p} \leq \left|z\right|^{k-p} < 1.$$

This completes the proof of the theorem. □

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Birgül Öner and Sevtap Sümer Eker
Dicle University, Department of Mathematics
Science Faculty
TR-21280 Diyarbakır, Turkey
e-mail: sevtaps@dicle.edu.tr