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On certain subclasses of meromorphic functions defined by convolution with positive and fixed second coefficients

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Abstract. In this paper we consider the class $M(f,g;\alpha,\beta,\lambda,c)$ of meromorphic univalent functions defined by convolution with positive coefficients and fixed second coefficients. We obtained coefficient inequalities, distortion theorems, closure theorems, the radii of meromorphic starlikeness, and convexity for functions of this class.

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1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic in the punctured unit disc $U^*=\{z\in\mathbb{C}:0<|z|<1\}=U\backslash\{0\}$. Let $g\in\Sigma$, be given by

$$g(z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_kz^k, \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f*g)(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kb_kz^k=(g*f)(z). \tag{1.3}$$

A function $f\in\Sigma$ is meromorphically starlike of order β $(0\le\beta<1)$ if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\beta\ (z\in U), \tag{1.4}$$

the class of all such functions is denoted by $\Sigma^*(\beta)$. A function $f \in \Sigma$ is meromorphically convex of order β $(0 \leq \beta < 1)$ if

$$-\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in U), \tag{1.5}$$

the class of such functions is denoted by $\Sigma_k(\beta)$. The classes $\Sigma^*(\beta)$ and $\Sigma_k(\beta)$ were introduced and studied by Pommerenke [18], Miller [15], Mogra et al. [16], Cho [9], Cho et al. [10] and Aouf ([1] and [2]).
It is easy to observe from (1.4) and (1.5) that

$$f \in \Sigma_k(\beta) \Longleftrightarrow -zf' \in \Sigma^*(\beta).$$

For $\alpha \geq 0,\ 0 \leq \beta < 1,\ 0 \leq \lambda < \frac{1}{2}$ and g given by (1.2) with $b_k > 0$ $(k \geq 1)$, Aouf et al. [3] defined the class $M(f,g;\alpha,\beta,\lambda)$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\Re\left\{\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+\beta\right\}$$
$$\geq \alpha\left|\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+1\right| (z \in U). \tag{1.6}$$

We note that for suitable choices of $g,\ \alpha$ and λ, we obtain the following subclasses of the class $M(f,g;\alpha,\beta,\lambda)$:

(1) $M\left(f,\dfrac{1}{z(1-z)};0,\beta;0\right) = \Sigma^*(\beta)$ $(0 \leq \beta < 1)$ (see Pommerenke [18]);

(2) $M\left(f,\dfrac{1}{z}+\sum\limits_{k=1}^{\infty} D_k(\gamma)z^k;\alpha,\beta,\lambda\right) = \Sigma_\gamma(\alpha,\beta,\lambda)$ (see Atshan and Kulkarni [7] and Atshan [6]) $(\alpha \geq 0,\ 0 \leq \beta < 1,\ \gamma > -1,\ 0 \leq \lambda < \dfrac{1}{2})$, where

$$D_k(\gamma) = \frac{(\gamma+1)(\gamma+2)...(\gamma+k+1)}{(k+1)!}; \tag{1.7}$$

(3) $M\left(f,\dfrac{1}{z}+\sum\limits_{k=1}^{\infty} \Gamma_k(\alpha_1)z^k;\alpha,\beta,\lambda\right) = \Sigma(\beta,\alpha,\lambda)$ (see Magesh et al. [14]) $(\alpha \geq 0,$ $0 \leq \beta < 1, 0 \leq \lambda < \dfrac{1}{2})$, where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k+1}...(\alpha_q)_{k+1}}{(\beta_1)_{k+1}...(\beta_s)_{k+1}}\frac{1}{(k+1)!}; \tag{1.8}$$

(4) $M\left(f,\dfrac{1}{z}+\sum\limits_{k=1}^{\infty} \Gamma_k(\alpha_1)z^k;0,\beta,\lambda\right) = M_s^q(\lambda,\beta)$ (see Murugusundaramoorthy et al. [17]) $(0 \leq \beta < 1,\ 0 \leq \lambda < \dfrac{1}{2},\ q \leq s+1,\ q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1,2,...\})$, where $\Gamma_k(\alpha_1)$ is defined by (1.8).

Also, we note that

(1) $M(f,g;\alpha,\beta,0) = N(f,g;\alpha,\beta)$

$$= \left\{f \in \Sigma : -\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\beta\right) \geq \alpha\left|\frac{z(f*g)'(z)}{(f*g)(z)}+1\right|\right\} (z \in U); \tag{1.9}$$

(2) Putting $g(z)=\frac{1}{z}+\sum\limits_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^{m}z^{k}$ in (1.6), then the class

$$M\left(f,\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^{m}z^{k};\alpha,\beta,\lambda\right)$$

reduces to the class

$$M_{\delta,\ell}(m;\alpha,\beta,\lambda)=\left\{f\in\Sigma:-\Re\left\{\frac{z(I^{m}(\delta,\ell)f(z))'+\lambda z^{2}(I^{m}(\delta,\ell)f(z))''}{(1-\lambda)(I^{m}(\delta,\ell)f(z))+\lambda z(I^{m}(\delta,\ell)f(z))'}+\beta\right\}\geq\alpha\right.$$

$$\left.\left|\frac{z(I^{m}(\delta,\ell)f(z))'+\lambda z^{2}(I^{m}(\delta,\ell)f(z))''}{(1-\lambda)(I^{m}(\delta,\ell)f(z))+\lambda z(I^{m}(\delta,\ell)f(z))'}+1\right|\ (\delta\geq 0,\ \ell>0,\ m\in\mathbb{N}_{0},\ z\in U)\right\},$$

where the operator

$$I^{m}(\delta,\ell)(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^{m}z^{k},\tag{1.10}$$

was introduced and studied by Bulboacă et al. [8], El-Ashwah [11 with $p=1$] and El-Ashwah et al. [12 with $p=1$].

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0\leq\beta<1,\ 0\leq\lambda<\frac{1}{2},\ \alpha\geq 0$, g is given by (1.2) with $b_k>0$ and $b_k\geq b_1\,(k\geq 1)$.

We begin by recalling the following lemma due to Aouf et al. [4].

Lemma 1.1. *Let the function f be defined by (1.1). Then f is in the class $M(f,g;\alpha,\beta,\lambda)$ if and only if*

$$\sum_{k=1}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k a_k\leq(1-\beta)(1-2\lambda).\tag{1.11}$$

Proof. In view of (1.11), we can see that the functions f defined by (1.1) in the class $M(f,g;\alpha,\beta,\lambda)$ and satisfy the coefficient inequality

$$a_1\leq\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\,b_1}.\tag{1.12}$$

Hence we may take

$$a_1=\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1},\ 0<c<1.\tag{1.13}$$

Making use of (1.13), we now introduce the following class of functions:

Let $M(f,g;\alpha,\beta,\lambda,c)$ denote the subclass of $M(f,g;\alpha,\beta,\lambda)$ consisting of functions of the form:

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}z+\sum_{k=2}^{\infty}a_kz^k\ \ (a_k\geq 0;0<c<1).\tag{1.14}$$

Motivated by the works of Aouf and Darwish [3], Aouf and Joshi [5], Ghanim and Darus [13] and Uralegaddi [19], we now introduce the following class of meromorphic functions with fixed second coefficients.

2. Coefficient estimates

Theorem 2.1. *Let the function f be defined by* (1.14)*. Then f is in the class $M(f,g;\alpha,\beta,\lambda,c)$, if and only if,*

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k a_k \leq (1-\beta)(1-2\lambda)(1-c). \tag{2.1}$$

Proof. Putting

$$a_1=\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}, \qquad 0<c<1, \tag{2.2}$$

in (1.11) and simplifying we get the required result. The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}z+\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k}z^k,\, k\geq 2. \tag{2.3}$$

Corollary 2.1. *Let the function f defined by* (1.13) *be in the class $M(f,g;\alpha,\beta,\lambda,c)$, then*

$$a_k\leq\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k}, \qquad k\geq 2. \tag{2.4}$$

The result is sharp for the function f given by (2.3)*.*

3. Growth and Distortion theorems

Theorem 3.1. *If the function f defined by* (1.14) *is in the class $M(f,g;\alpha,\beta,\lambda,c)$ for $0<|z|=r<1$, then we have*

$$\frac{1}{r}-\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}r-\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)\,b_2}r^2\leq|f(z)|$$

$$\leq\frac{1}{r}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}r+\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)\,b_2}r^2. \tag{3.1}$$

The result is sharp for the function f given by

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}z+\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)\,b_2}z^2. \tag{3.2}$$

Proof. Since $f\in M(f,g;\alpha,\beta,\lambda,c)$, then Theorem 2.1 yields

$$a_k\leq\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k}, \qquad k\geq 2. \tag{3.3}$$

Thus, for $0<|z|=r<1$,

$$|f(z)|\leq\frac{1}{|z|}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}|z|+\sum_{k=2}^{\infty}a_k|z|^k$$

$$\leq\frac{1}{r}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}r+r^2\sum_{k=2}^{\infty}a_k$$

$$\leq \frac{1}{r} + \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} r + \frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2} r^2, \text{ by } (3.3).$$

Also we have

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} |z| - \sum_{k=2}^{\infty} a_k |z|^k \\ &\geq \frac{1}{r} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} r - r^2 \sum_{k=2}^{\infty} a_k \\ &\geq \frac{1}{r} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} r - \frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2} r^2. \end{aligned}$$

Thus the proof of Theorem 3.1 is completed.

Theorem 3.2. *If the function f defined by* (1.14) *is in the class $M(f, g; \alpha, \beta, \lambda, c)$ for $0 < |z| = r < 1$, then we have*

$$\begin{aligned} &\frac{1}{r^2} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} - \frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2} r \\ &\leq \left|f'(z)\right| \leq \frac{1}{r^2} + \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} + \frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2} r. \end{aligned} \tag{3.4}$$

The result is sharp for the function f given by (3.2).

Proof. In view of Theorem 2.1, it follows that

$$ka_k \leq \frac{k(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}, \qquad k \geq 2. \tag{3.5}$$

Thus, for $0 < |z| = r < 1$, and making use of (3.5), we obtain

$$\begin{aligned} \left|f'(z)\right| &\leq \frac{1}{|z^2|} + \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} + \sum_{k=2}^{\infty} ka_k |z|^{k-1} \\ &\leq \frac{1}{r^2} + \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} + r\sum_{k=2}^{\infty} ka_k \\ &\leq \frac{1}{r^2} + \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} + \frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2} r, \text{ by } (3.5). \end{aligned}$$

Also we have

$$\begin{aligned} \left|f'(z)\right| &\geq \frac{1}{|z^2|} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} - \sum_{k=2}^{\infty} ka_k |z|^{k-1} \\ &\geq \frac{1}{r^2} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} - r\sum_{k=2}^{\infty} ka_k \\ &\geq \frac{1}{r^2} - \frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1} - \frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2} r. \end{aligned}$$

Hence the result follows.

4. Closure theorems

In this section we shall show that the class $M(f,g;\alpha,\beta,\lambda,c)$ is closed under convex linear combination.

Theorem 4.1. *Let*

$$f_1(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z, \tag{4.1}$$

and

$$f_k(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+ \tag{4.2}$$
$$\sum_{k=2}^{\infty}\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}z^k \qquad (k\geq 2).$$

Then $f\in M(f,g;\alpha,\beta,\lambda,c)$, if and only if it can expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z), \tag{4.3}$$

where $\mu_k\geq 0$ and $\sum\limits_{k=1}^{\infty}\mu_k\leq 1$.

Proof. Let

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z),$$

then from (4.2) and (4.3), we have

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\sum_{k=2}^{\infty}\frac{(1-\beta)(1-2\lambda)(1-c)\mu_k}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}z^k. \tag{4.4}$$

Since

$$\sum_{k=2}^{\infty}\frac{(1-\beta)(1-2\lambda)(1-c)\mu_k}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}{(1-\beta)(1-2\lambda)(1-c)}$$
$$=\sum_{k=2}^{\infty}\mu_k=1-\mu_1\leq 1,$$

hence by using Lemma 1.1, we have $f\in M(f,g;\alpha,\beta,\lambda,c)$.
Conversely, suppose that f defined by (1.14) is in the class $M(f,g;\alpha,\beta,\lambda,c)$. Then by using (2.4) ,we get

$$a_k\leq\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}, \qquad k\geq 2. \tag{4.5}$$

Setting

$$\mu_k=\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}{(1-\beta)(1-2\lambda)(1-c)}, \quad k\geq 2 \tag{4.6}$$

and

$$\mu_1=1-\sum_{k=2}^{\infty}\mu_k, \tag{4.7}$$

we can see that f can be expressed in the form (4.3) .This completes the proof of Theorem 4.1.

Theorem 4.2. *The class $M(f,g;\alpha,\beta,\lambda,c)$ is closed under linear combination.*

Proof. Suppose that the function f given by (1.14) , and the function g given by

$$g(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\sum_{k=2}^{\infty}d_kz^k,\qquad d_k\geq 0. \tag{4.8}$$

Assuming that f and g are in the class $M(f,g;\alpha,\beta,\lambda,c)$, it is enough to prove that the function h defined by

$$h(z)=\mu f(z)+(1-\mu)g(z),\qquad 0\leq\mu\leq 1, \tag{4.9}$$

is also in the class $M(f,g;\alpha,\beta,\lambda,c)$. Since

$$h(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\sum_{k=2}^{\infty}[a_k\mu+(1-\mu)d_k]z^k, \tag{4.10}$$

we observe that

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k[a_k\mu+(1-\mu)d_k]$$
$$\leq(1-\beta)(1-2\lambda)(1-c), \tag{4.11}$$

with the aid of Theorem 2.1. Thus, $h\in M(f,g;\alpha,\beta,\lambda,c)$.

5. Radii of Meromorphically Starlikeness and Convexity

Theorem 5.1. *Let the function f defined by (1.14) be in the class $M(f,g;\alpha,\beta,\lambda,c)$. Then f is meromorphically starlike of order $\delta\,(0\leq\delta<1)$ in $0<|z|<r_1(\alpha,\beta,\lambda,c,\delta)$, where $r_1(\alpha,\beta,\lambda,c,\delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\frac{(k+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}r^{k+1}\leq(1-\delta), \tag{5.1}$$

for $k\geq 2$. The result is sharp for the function

$$f_k(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}z^k \tag{5.2}$$

for some k.

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\leq 1-\delta\,(0\leq\delta<1)\ \text{ for } 0<|z|<r_1. \tag{5.3}$$

Note that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\leq\frac{\frac{2(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\sum\limits_{k=2}^{\infty}(k+1)a_kr^{k+1}}{1-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2-\sum\limits_{k=2}^{\infty}a_kr^{k+1}}\leq 1-\delta \tag{5.4}$$

for $(0 \leq \delta < 1)$ if and only if

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\sum_{k=2}^{\infty}(k+2-\delta)a_k r^{k+1}\leq(1-\delta). \tag{5.5}$$

Since f is in the class $M(f,g;\alpha,\beta,\lambda,c)$, from (2.4), we may take

$$a_k=\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}\mu_k \qquad (k\geq 2), \tag{5.6}$$

where $\mu_k \geq 0\,(k\geq 2)$ and $\sum\limits_{k=2}^{\infty}\mu_k \leq 1$.

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which

$$\frac{(k+2-\delta)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]}r^{k+1}$$

is maximal. Then it follows that

$$\sum_{k=2}^{\infty}(k+2-\delta)a_k r^{k+1}\leq\frac{(k_0+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k_0-1)][k_0(1+\alpha)+(\alpha+\beta)]b_{k_0}}r^{k_0+1}. \tag{5.7}$$

Then f is starlike of order δ in $0<|z|<r_1(\alpha,\beta,\lambda,c,\delta)$ provided that

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\frac{(k_0+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k_0-1)][k_0(1+\alpha)+(\alpha+\beta)]b_{k_0}}r^{k_0+1}\leq(1-\delta). \tag{5.8}$$

We find the value $r_0=r_0(\alpha,\beta,\lambda,c,\delta)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r_0^2+\frac{(k_0+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k_0-1)][k_0(1+\alpha)+(\alpha+\beta)]b_{k_0}}r_0^{k_0+1}=(1-\delta). \tag{5.9}$$

Then this value r_0 is the radius of meromorphically starlike of order δ for functions belonging to the class $M(f,g;\alpha,\beta,\lambda,c)$.

Corollary 5.1. *Let the function f defined by* (1.14) *be in the class $M(f,g;\alpha,\beta,\lambda,c)$. Then f is meromorphically convex of order $\delta\,(0\leq\delta<1)$ in $0<|z|<r_2(\alpha,\beta,\lambda,c,\delta)$, where $r_2(\alpha,\beta,\lambda,c,\delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\frac{k(k+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}r^{k+1}\leq(1-\delta), \tag{5.10}$$

$(k\geq 2)$. *The result is sharp for function f given by* (5.2) *for some k.*

Remark. Specializing the function g, in(1.6), we have results for the subclasses maintain in the introduction in the case of fixed second coefficients.

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