

Finding initial solutions for a class of nonlinear BVP

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Abstract. The purpose of these paper is to solve a nonlinear boundary value problem having the origin in fluid mechanics. The equation has in general several solutions and the main difficulty is to find starting solutions. We follow a mixed symbolic-numeric approach.

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1. Introduction

1.1. The origin of the problem

Consider the steady stagnation-point flow of a viscous and incompressible fluid over a continuously stretching/shrinking sheet in its own plane.

Assumptions [4]:

- free stream velocity $u_e(x) = ax$, stretching/shrinking sheet velocity $u_w(x) = b(x + c) + u_{slip}(x)$, where $b > 0$ is the stretching rate and $b < 0$ is the shrinking rate, $a > 0$, $-c$ is the location of the stretching origin and is $u_{slip}(x)$ the second-order velocity slip
- x -axis is measured along the stretching surface and the y -axis is perpendicular to it

The basic equations are [1, 3]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (1.2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \quad (1.3)$$

where u and v are the velocity components along the x - and y -axes, p is the pressure, ν is the kinematic viscosity and ρ is the density.

For the previous equations, we have the following boundary conditions

$$\begin{cases} v = v_0, & u = u_w(x) = b(x + c) + u_{slip}(x) & \text{at } y = 0 \\ u = u_e(x) = ax & & \text{as } y \rightarrow \infty \end{cases} \quad (1.4)$$

where v_0 is the mass flux velocity $v_0 < 0$ with for suction and $v_0 > 0$ for injection. [5] and [1] give

$$u_{slip}(x) = A \frac{\partial u}{\partial y} + B \frac{\partial^2 u}{\partial y^2}, \quad A \in \mathbb{R}, B < 0. \quad (1.5)$$

Following [4], we assume that Eqs. (1.1) to (1.3) subject to the boundary conditions (1.4) admit the similarity solution

$$u = axf'(\eta) + bcg(\eta), \quad v = \sqrt{av}f(\eta), \quad \eta = y\sqrt{\frac{a}{\nu}}. \quad (1.6)$$

where prime ($'$) denotes differentiation with respect to η .

Using Eq. (1.3) and the boundary conditions (1.5), we obtain the following expression for the pressure p

$$p = p_0 - \rho \frac{u_e(x)^2}{2} - \mu \frac{v^2}{2} + \mu \frac{\partial v}{\partial y} \quad (1.7)$$

where p_0 is the stagnation pressure. Thus, we have

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = u_e \frac{\partial u_e}{\partial x} \quad (1.8)$$

1.2. The boundary value problem

Substituting (1.6) and (1.8) into Eq. (1.2), we get the system of ODE

$$f''' + ff'' + f'^2 - 1 = 0 \quad (1.9)$$

$$g'' + fg' - f'g = 0 \quad (1.10)$$

subject to the boundary conditions

$$f(0) = s, \quad f'(0) = \alpha + \beta f''(0) + \gamma f'''(0), \quad f'(\infty) = 1 \quad (1.11)$$

$$g(0) = 1, \quad g'(\infty) = 0. \quad (1.12)$$

Here $s = -v_0/\sqrt{av}$ is the mass flux velocity with $s > 0$ for suction and $s < 0$ for injection, respectively, $\alpha > 0$ is the stretching parameter and $\alpha < 0$ is the shrinking parameter, respectively, β is the first order velocity slip parameter with $0 < \beta = a\sqrt{c/\nu}$ and γ is the second order velocity slip parameter with $\gamma = Bc/\nu < 0$.

2. The symbolic solution

Once f is determined, the second equation (1.10) is linear in g ; so, we focus our attention to the solution of (1.9) + (1.11).

The next Maple code shows the result returned by Maple procedure `dsolve` for (1.9).

```
> ecd1:=diff(f(x),x$3)+f(x)*diff(f(x),x$2)+diff(f(x),x)^2-1=0;
```

$$ecd1 := \frac{d^3}{dx^3}f(x) + f(x) \frac{d^2}{dx^2}f(x) + \left(\frac{d}{dx}f(x)\right)^2 - 1 = 0$$

```
> sed:=dsolve(ecd1,f(x));
```

$$\begin{aligned} f(x) = & \frac{1}{6} ((-C1^4 - C3 - 2C1^3 - C3x + C1^2 - C3x^2 \\ & + 2C1^2 - C2 - C3 - 4C1 - C2 - C3x + 2C2 - C3x^2 - 6C1^2 - C3 \\ & + 12C1 - C3x - 6C3x^2) \operatorname{hypergeom}\left(\left[\frac{7}{4} - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \right. \\ & \left. \left[\frac{5}{2}\right], \frac{1}{2}(-C1 - x^2)\right) / ((-C1 - C3 - C3x) \operatorname{hypergeom}\left(\left[\frac{3}{4} \right. \right. \\ & \left. \left. - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(-C1 - x^2)\right) + \operatorname{hypergeom}\left(\left[\frac{1}{4} \right. \right. \\ & \left. \left. - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{1}{2}\right], \frac{1}{2}(-C1 - x^2)\right) + \frac{1}{6} ((3C1^3 \\ & - 3C1^2x + 6C1 - C2 - 6C2x - 6C1 + 6x) \operatorname{hypergeom}\left(\left[\frac{5}{4} \right. \right. \\ & \left. \left. - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(-C1 - x^2)\right)) / ((-C1 - C3 \\ & - C3x) \operatorname{hypergeom}\left(\left[\frac{3}{4} - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(-C1 - x^2)\right) \\ & + \operatorname{hypergeom}\left(\left[\frac{1}{4} - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{1}{2}\right], \frac{1}{2}(-C1 - x^2)\right) \\ & + \frac{1}{6} ((6C1^2 - C3 - 12C1 - C3x + 6C3x^2 - 12C3) \\ & \operatorname{hypergeom}\left(\left[\frac{3}{4} - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(-C1 - x^2)\right) \\ & + (6C1 - 6x) \operatorname{hypergeom}\left(\left[\frac{1}{4} - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{1}{2}\right], \right. \\ & \left. \frac{1}{2}(-C1 - x^2)\right) / ((-C1 - C3 - C3x) \\ & \operatorname{hypergeom}\left(\left[\frac{3}{4} - \frac{1}{4}C2 - \frac{1}{8}C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(-C1 - x^2)\right) + \end{aligned}$$

$$\text{hypergeom} \left(\left[\left[\frac{3}{4} - \frac{1}{4} _C2 - \frac{1}{8} _C1^2 \right], \left[\frac{3}{2} \right], \frac{1}{2} (_C1 - x^2) \right] \right)$$

Here $\text{hypergeom}(a, b, z)$ means the confluent hypergeometric function ${}_1F_1(a, b, z)$ (see [2])

$${}_1F_1(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_{2s!}} z^s = 1 + \frac{a}{b} z + \frac{a(a+1)}{2b(b+1)} z^2 + \dots$$

The solution depends on three free constants $_C1, _C2, _C3$. (1.11) leads us to a nonlinear system in unknowns $_C1, _C2, _C3$:

$$\begin{aligned} F(0) &= s \\ F'(0) &= \alpha + \beta F''(0) + \gamma F'''(0), \\ F'(\infty) &= 1 \end{aligned} \tag{2.1}$$

3. The numerical solution

Our approach consists of:

- Replace ${}_1F_1(a, b, z)$ in the expression of F by an approximation and solve the system
- Use the solution as a starting function for a numerical method for the BVP (1.9)+(1.10)+(1.11)+1.12).

When z is close to 0 we use a Taylor expansion with a small number of terms

$$\begin{aligned} {}_1F_1(a, b, z) &\approx \sum_{s=0}^n \frac{(a)_s}{(b)_{2s!}} z^s \\ &= 1 + \frac{a}{b} z + \frac{a(a+1)}{2b(b+1)} z^2 + \dots + \frac{a(a+1) \cdots (a+n-1)}{n!b(b+1) \cdots (b+n-1)} z^n \end{aligned}$$

If z is large,

$${}_1F_1(a, b, z) \approx \frac{e^z z^{a-b} \Gamma(b)}{\Gamma(a)}.$$

(see [2]).

With hypergeometric function replaced by previous approximation, Maple procedure `solve` returns the approximate solution of (2.1) in terms of s, α, β, γ . This approach allows us to obtain informations about the number of solutions and their behavior.

Finally, we solve BVP (1.9)+(1.10)+(1.11)+1.12) using the MATLAB function `bvp4c` and the approximation of F as the starting function (initial solution).

3.1. Numerical examples

We consider two numerical examples.

- For $s = -1$, $\alpha = 2$, $\beta = 0.5$, $\gamma = -1$, Maple returns two solutions

$$\begin{aligned} &.986091543197571, -3.48618826578288, &&-.848486949052433, \\ &2.29686508693243, -5.63779461378456, &&-1.16744311525857 \end{aligned}$$

Using these coefficients for the MATLAB starting solution, we obtain the graph in Figure 1.

- For $s = -1$, $\alpha = 2$, $\beta = 0.5$, $\gamma = -1$, Maple returns four solutions

$$\begin{aligned} &2.17791394594416, -5.37165457796903, &&-1.18215998400672, \\ &-.497833460462187, -3.12391907717788, &&.959680066577520, \\ &-.654766323087723, -3.21435946892491, &&1.04272129118383, \\ &-1.13575318603863, -3.64496764979845, &&1.16765563652685 \end{aligned}$$

Using these coefficients for the MATLAB starting solution, we obtain the graph in Figure 2.

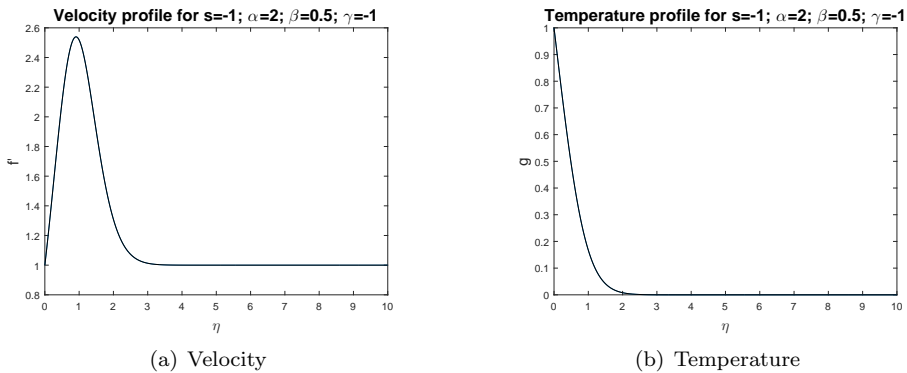


FIGURE 1. Velocity (left) and temperature profile for the first example

We give in the sequel the Maple code for the computation of initial solutions

```
> restart;
> Digits:=15:
A Taylor expansion of the hypergeometric function
> hyge:=proc(a,b,z,n::integer:=2)
> add(z^k*factor(expand(pochhammer(op(a),k)))/
> (k!*factor(expand(pochhammer(op(b),k))))),k=0..n);
> end:
Equations and boundary conditions
> ecd1:=diff(f(x),x$3)+f(x)*diff(f(x),x$2)+diff(f(x),x)^2-1=0:
```

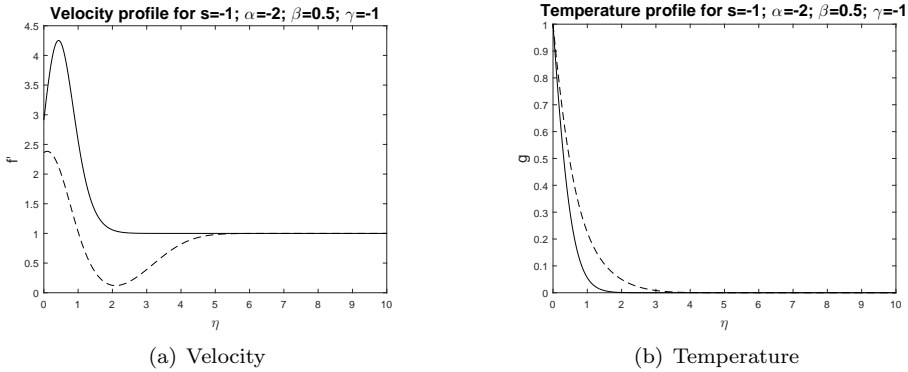


FIGURE 2. Velocity (left) and temperature profile for the second example

```

> ic:=f(0)=s,
> D(f)(0)=alpha+beta*D[1,1](f)(0)+Gamma*D[1,1,1](f)(0),
> D(f)(10)=1:

> sed:=dsolve(ecd1,f(x)):
This is the "analytic" solution!

> F:=unapply(rhs(%),x):
Fh is an approximation of F.

> Fh:=unapply(subs(hypergeom=hyge,F(x)),x):
> Fh:=unapply(simplify(Fh(x)),x):
The nonlinear system to be solved

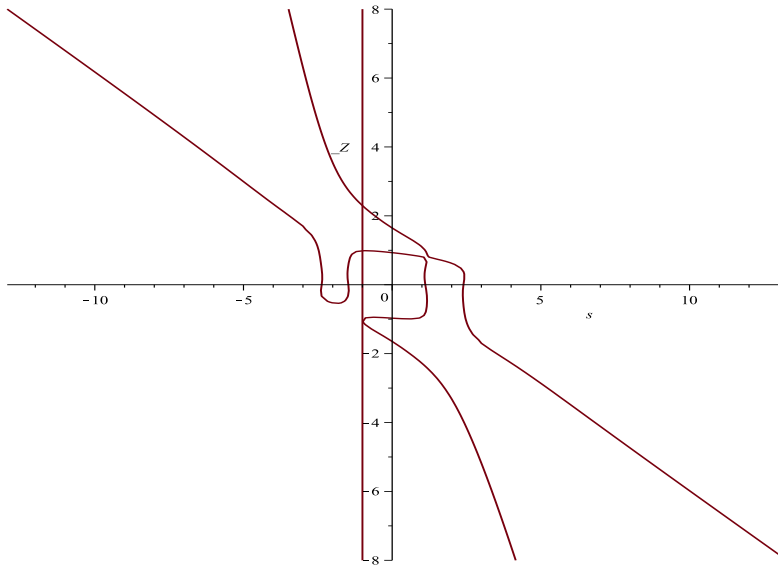
> ec1:=Fh(0)=s:
> ec2:=eval(diff(Fh(x),x),x=0)=alpha+beta*eval(diff(Fh(x),x,x),x=0)
> +Gamma*eval(diff(Fh(x),x$3),x=0):
> ec3:=limit(diff(Fh(x),x),x=infinity)=1:

> with(plots):

> ss:=solve([ec1,ec2,ec3],[_C1,_C2,_C3]):
> vss:=eval(ss,[alpha=2, beta=0.5, Gamma=-1]):
> assign(%);

> implicitplot([op(_C1),s=-1],s=-15..15,_Z=-8..8,gridrefine=2,
crossingrefine=2);

```

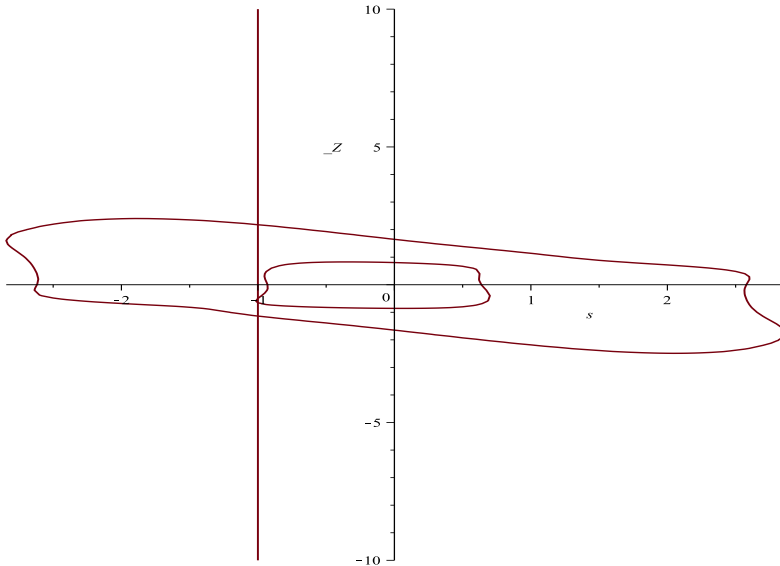


```
> unassign('_C1','_C2','_C3');
> vss1:=eval(ss,[s=-1,alpha=2, beta=0.5, Gamma=-1]):
> lv:=evalf(allvalues(vss1)):
> la:=map(op,[lv]):
> la2:=remove(has,la,I);
```

la2 :=

```
[[_C1 = .986091543197571,_C2 = -3.48618826578288,_C3 = -.848486949052433],
[_C1 = 2.29686508693243,_C2 = -5.63779461378456,_C3 = -1.16744311525857]]
```

```
> unassign('_C1','_C2','_C3');
> vssa:=eval(ss,[alpha=-2, beta=0.5, Gamma=-1]):
> assign(%);
> implicitplot([op(_C1),s=-1],s=-5..5,_Z=-10..10,gridrefine=2,
crossingrefine=2);
```



```

> unassign('_C1', '_C2', '_C3');
> vss1:=eval(ss, [s=-1, alpha=-2, beta=0.5, Gamma=-1]);
> lv:=evalf(allvalues(vss1));
> la:=map(op, [lv]);
> la2:=remove(has, la, I);

```

```
la2 :=
```

```

[[_C1 = 2.17791394594416, _C2 = -5.37165457796903, _C3 = -1.18215998400672],
[_C1 = -.497833460462187, _C2 = -3.12391907717788, _C3 = .959680066577520],
[_C1 = -.654766323087723, _C2 = -3.21435946892491, _C3 = 1.04272129118383],
[_C1 = -1.13575318603863, _C2 = -3.64496764979845, _C3 = 1.16765563652685]]

```

References

- [1] Fang, T., Yao, S., Zhang, J., Aziz, A., *Viscous flow over a shrinking sheet with a second order slip flow model*, Commun. Nonlinear Sci. Numer. Simulat., **15**(2010), 1831–1842.
- [2] Frank W.J. Olver (Editor in chief), *NIST Handbook of Mathematical Functions*, NIST and Cambridge University Press, 2010.
- [3] Ishak, A., Lok, Y.Y., Pop, I., *Stagnation-point flow over a shrinking sheet in a micropolar fluid*, Chem. Engng. Communicat., **197**(2010), 1417-1427.
- [4] Wang, C.Y., *Stagnation flow towards a shrinking sheet*, Int. J. Non-Linear Mech., **43**(2008), 377-382.

- [5] Wu, L., *A slip model for rarefied gas flows at arbitrary Knudsen number*, Appl. Phys. Lett., **93**(2008).
- [6] *Maple 18 User Manual*, Maplesoft, 2014.
- [7] *Maple 18 Programming Guide*, Maplesoft, 2014.
- [8] *Mathematics in MATLAB*, Mathworks Inc., 2014.

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