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On the unbounded divergence of interpolatory product quadrature rules on Jacobi nodes

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Abstract. This paper is devoted to prove the unbounded divergence on superdense sets, with respect to product quadrature formulas of interpolatory type on Jacobi nodes.

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1. Introduction

Let μ be the Lebesgue measure on the interval $[-1,1]$ of $\mathbb{R}$ and let denote by L_1 the Banach space of all measurable functions (equivalence classes of functions with respect to the equality $\mu-$a.e.) $g : [-1,1] \to \mathbb{R}$, such that $|g|$ is Lebesgue integrable on the interval $[-1,1]$, endowed with the norm $||g||_1 = \int_{-1}^{1} |g(x)|dx, g \in L_1$. Analogously, L_∞ is the Banach space of all measurable functions (equivalence classes of functions with respect to the equality $\mu-$ a.e) $g : [-1,1] \to \mathbb{R}$, normed by $||g||_\infty = \text{ess sup}\,|g|$.

Given a nonnegative function $\rho \in L_\infty$ such that $\rho(x) > 0$ μ -a.e. on [-1, 1], let consider, in accordance with [8], [9], the Banach space $(L_1^{(1/\rho)}, ||.||_1^{(1/\rho)})$, where $L_1^{(1/\rho)}$ is the set of all measurable functions (classes of functions) $g : [-1,1] \to \mathbb{R}$ for which $g/\rho \in L_1$ and $||g||^{(1/\rho)} = ||g/\rho||_1$.

Further, let denote by $(C, ||.||)$ the Banach space of all continuous functions $f : [-1,1] \to \mathbb{R}$, where $||.||$ stands for the uniform (supremum) norm, and let consider the Banach space $(C^s, ||.||_s)$ of all functions $f : [-1,1] \to \mathbb{R}$, that are continuous together with their derivatives up to the order $s \geq 1$, endowed with the norm

$$||f||_s = \sum_{r=0}^{s-1} |f^{(r)}(0)| + ||f^{(s)}||;$$

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we admit $f^{(0)} = f$ and $C^0 = C$.

For each integer $n \geq 1$, let denote by $x_n^k = \cos\ \theta_n^k, 1 \leq k \leq n, 0 < \theta_n^1 < \theta_n^2 < \ldots < \theta_n^n < \pi$, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$, with $\alpha > -1$ and $\beta > -1$, referred to as Jacobi nodes.

We specify, also, the usual notations

$$(L_n f)(x) = \sum_{k=1}^{n} f(x_n^k) l_n^k(x),\ |x| \leq 1$$

and

$$\Lambda_n(x) = \sum_{k=1}^{n} |l_n^k(x)|;\ |x| \leq 1,\ n \geq 1,$$

denoting the Lagrange polynomials which interpolate a function $f\colon [-1,1] \to \mathbb{R}$ at the Jacobi nodes, and the Lebesgue functions associated to the Jacobi nodes, respectively.

In this paper, we deal with *product-quadrature formulas* of interpolatory type, as follows:

$$I(f;g) = I_n(f;g) + R_n(f;g), n \geq 1, f \in C, g \in L_1^{(1/\rho)}, \tag{1.1}$$

where

$$I : C \times L_1^{(1/\rho)} \longrightarrow \mathbb{R}\ I(f;g) = \int_{-1}^{1} f(x)g(x)dx, \tag{1.2}$$

and

$$I_n(f;g) = \int_{-1}^{1} (L_n f)(x)g(x)dx, n \geq 1; f \in C, g \in L_1^{(1/\rho)}. \tag{1.3}$$

Numerous papers have studied the convergence of the product quadrature formulas of type (1.1), involving Jacobi, Gauss-Kronrod or equidistant nodes and various functions $g \in L_1$ (i.e. $\rho(x) = 1,\ \forall\ x \in [-1,1]$), [1, Ch. 5], [3], [4], [5], [7], [8], [9]. Regarding the divergence of these formulas, I.H.Sloan and W.E.Smith, [9, Th.7, ii] proved the following statement in the case of *Jacobi nodes:*

If $\alpha > -1$, $\beta > -1$ *and* $\rho(x) = (1-x)^{\max\{0,(2\alpha+1)/4\}}(1+x)^{\max\{0,(2\beta+1)/4\}}$, *then there exist a function* $f_0 \in C$ *and a function* $g_0 \in L_1^{(1/\rho)}$ *so that the sequence* $I_n(f_0, g_0) : n \geq 1$ *is not convergent to* $I(f_0, g_0)$.

In fact, the divergence phenomenon holds on large subsets of $L_1^{(1/\rho)}$ and C, in topological sense. More exactly, the following assertion is a particular case of [6, Theorem 3.2]:

Suppose that $\mu\{x \in [-1,1] : \rho(x) > 0\} > 0$. *Then, there exists a superdense set* X_0 *in the Banach space* $L_1^{(1/\rho)}$ *such that for every* g *in* X_0 *the subset of* C *consisting of all functions* f *for which the product integration rules (1.1) unboundedly diverge, namely*

$$Y_0(g) = \left\{ f \in C : \sup\left\{ \left| \int_{-1}^{1} (L_n f)(x)g(x)dx \right| ; n \geq 1 \right\} = \infty \right\},$$

is superdense in the Banach space C.

We recall that a subset S of the topological space T is said to be *superdense* in T if it is residual (namely its complement is of first Baire category), uncountable and dense in T.

The aim of this paper is to highlight the phenomenon of double condensation of singularities for the product quadratures formulas (1.1) in the case of the Banach spaces $(C^s, ||.||_s)$, $s \geq 1$. If $\rho(x) = 1$, $\forall\ x \in [-1, 1]$, and $\alpha = \beta = 2$, this property was emphasized in [5, Th.3], for $s = 1$ and $s = 2$. In the next section, we point out the double superdense unbounded divergence of the formulas (1.1) for $\alpha > -1$, $\beta > -1$ and $s \geq 1$ satisfying the inequality $s < \alpha + 1/2$ or $s < \beta + 1/2$ and more general conditions regarding the function ρ.

In what follows, we denote by m, M, M_k, $k \geq 1$, some generic positive constants which are independent of any positive integer n and we use the notation $a_n \sim b_n$ if the sequences (a_n) and (b_n) satisfy the inequalities $0 < m \leq |a_n/b_n| \leq M$.

2. The unbounded divergence of the product quadrature formulas (1.1)

Let $T_n f : C^s \to (L_1^{(1/\rho)})^*$, be the continuous linear operators given by $T_n f : L_1^{(1/\rho)} \to \mathbb{R}, f \in C^s$ and $(T_n f)(g) = \int_{-1}^{1} g(x)(L_n f)(x)dx,\ g \in L_1^{(1/\rho)}$ $n \geq 1$, where $(L_1^{(1/\rho)})^*$ is the Banach space of all continuous linear functionals defined on $L_1^{(1/\rho)}$.

By standard reasoning, via the Theorem of Riesz concerning the representation of continuous linear functionals, we get:

$$||T_n|| = \sup\{||\rho L_n f||_\infty : f \in C^s, ||f||_s \leq 1\}. \tag{2.1}$$

Now, we are in the position to state the following divergence result:

Theorem 2.1. *Suppose that the integer $s \geq 0$ and the real numbers $A > 0$, $a \in (0, 1)$, $\alpha > -1$, $\beta > -1$ satisfy at least one of the following conditions:*
(i) $s < \alpha + 1/2$ and $\rho(x) \geq A$, for $x \in (a, 1)$;
(ii) $s < \alpha + 1/2$ and $\rho(1) > 0$;
(iii) $s < \beta + 1/2$ and $\rho(x) \geq A$, for $x \in (-1, -a)$;
(iv) $s < \beta + 1/2$ and $\rho(-1) > 0$.

Then, there exists a superdense set X_0 in the Banach space $L_1^{(1/\rho)}$, such that for every g in X_0 the subset of C^s consisting of all functions f for which the product integration rules (1.1) unboundedly diverge, namely

$$Y_0(g) = \left\{ f \in C^s : \sup \left\{ \left| \int_{-1}^{1} (L_n f)(x) g(x) dx \right| ; n \geq 1 \right\} = \infty \right\},$$

is superdense in the Banach space C^s.

Proof. For each integer $n \geq 2$, let us define the numbers $\delta_n^k, 1 \leq k \leq n$, and δ_n as follows: $3\delta_n^k = \min\{x_n^{k-1} - x_n^k, x_n^k - x_n^{k+1}\}$, $1 \leq k \leq n$, with $x_n^0 = 1, x_n^{n+1} = -1$, and $\delta_n = \max\{\delta_n^k, 1 \leq k \leq n\}$.

In analogy with [5, Th.2.3], we obtain:

$$||T_n|| \geq M_1\ \frac{\rho(\tau_n)}{(\delta_n)^{s+2}} \sum_{k=1}^{n} (\delta_n^k)^{2s+2} |l_n^k(\tau_n)|, \tag{2.2}$$

where τ_n is an arbitrary number of $[-1, 1]$.

For the beginning, let us suppose that the hypothesis (i) of this theorem is satisfied. The estimate sin $\theta_n^k \sim k/n$, [7], implies

$$\theta_n^k \sim k/n. \tag{2.3}$$

The relations $P_n^{(\alpha,\beta)}(x_n^1) = 0$ and $P_n^{(\alpha,\beta)}(1) \sim n^\alpha$, [10], lead to the existence of a point τ_n so that

$$\tau_n \in (x_n^1, 1);\ P_n^{(\alpha,\beta)}(\tau_n) = (1/2)P_n^{(\alpha,\beta)}(1) \sim n^\alpha. \tag{2.4}$$

Now, let us estimate δ_n^k, δ_n and $|l_n^k(\tau_n)|$.
The estimates $\theta_n^k - \theta_n^{k-1} \sim 1/n$, sin $\theta_n^k \sim k/n$ and $\theta \sim \theta_n^k$, if $\theta_n^{k-1} \le \theta \le \theta_n^k$, [7] , combined with $x_n^{k-1} - x_n^k = 2\sin(\theta_n^k - \theta_n^{k-1})/2 \sin(\theta_n^k + \theta_n^{k+1})/2$, yield:

$$\delta_n^k \sim k/n^2,\ 1 \le k \le n;\ \delta_n \sim 1/n. \tag{2.5}$$

The relation $\tau_n \in (x_n^1, 1)$ of (2.4), together with (2.3) and $x_n^1 \ge x_n^k$, $1 \le k \le n$, gives $|\tau_n - x_n^k| = \tau_n - x_n^k \le 1 - x_n^k = 2\sin^2(\theta_n^k/2) \sim k^2/n^2$, namely

$$|\tau_n - x_n^k| \le M_2 k^2/n^2, 1 \le k \le n. \tag{2.6}$$

Now, by combining the inequality (2.6) with the estimates (2.4) and $|(P_n^{(\alpha,\beta)}(x_n^k))'| \sim n^{\alpha+2}k^{-\alpha-3/2}$, if $0 < \theta_n^k < \pi/2$, [10], we get:

$$|l_n^k(\tau_n)| = |P_n^{(\alpha,\beta)}(\tau_n)||\tau_n - x_n^k|^{-1}|(P_n^{(\alpha,\beta)}(x_n^k))'|^{-1} \ge M_2 k^{\alpha-1/2}. \tag{2.7}$$

Further, the relation (2.3) with $k = 1$, together with (2.4), implies $\tau_n \in (a, 1)$, for n sufficiently large, which leads to:

$$\rho(\tau_n) \ge A > 0. \tag{2.8}$$

Finally, the relations (2.2), (2.4), (2.7) and (2.8) provide the inequality

$$||T_n|| \ge M_4 n^{\alpha+1/2-s}, \tag{2.9}$$

for n sufficiently large. Secondly, if the condition (ii) is fulfilled, we proceed in a similar manner, taking τ_n =1 in (2.2) and obtaining the unboundedness of the set of norms $\{||T_n|| : n \ge 1\}$ from an analogous inequality of (2.9). Also, it is easily seen that the hypotheses (iii) and (iv) lead to an inequality of type (2.9), namely:

$$||T_n|| \ge M_4 n^{\beta+1/2-s}, \tag{2.10}$$

for n sufficiently large.

To complete the proof, we apply, in a standard manner, firstly the principle of condensation of singularities, [2,Th.5.4], and the relations (2.9) and (2.10), in order to conclude that the set of unbounded divergence of the family $\{T_n : n \ge 1\}$ is superdense in the Banach spaces $(C^s, ||.||_s)$ and secondly, based on this result, the principle of double condensation of singularities, [2, Th.5.2], to provide the conclusion of this theorem. □

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