Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 2, 173–181

Estimates for the ratio of gamma functions by using higher order roots

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Abstract. It is the aim of this paper to give a systematically way for obtaining higher order roots estimates of the ratio $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$, as $x \to \infty$ and the Wallis ratio $\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$, as $n \to \infty$.

Mathematics Subject Classification (2010): 26D15, 11Y25, 41A25, 34E05.

Keywords: Gamma function, approximations, asymptotic series.

1. Introduction

The factorial function $n! = 1 \cdot 2 \cdot 3 \cdots n$ (defined for positive integers n), and its extension gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

(to the real and complex values z, excepting $-1, -2, -3, ...$) has a great importance in pure mathematics, as in applied mathematics and other branches of science, such as chemistry, statistical physics, or cuantum mechanics.

The ratio $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$ is strongly related to the Wallis sequence

$$P_n = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}$$

and to other aspects in the theory of the gamma function, as for example Kershaw-Gautschi inequalities. For this reason, many mathematicians have been preocuppied by the approximation of this ratio. There exists a broad literature on this subject. In particular, many inequalities, sharp bounds for these functions, and accurate approximations have been published. See, e.g. the classical results from [2] and the recent

This paper was presented at the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), Cluj-Napoca, Romania, September 17-20, 2014.

article [3] and all references therein. A first result was stated by Kazarinoff [4, pp. 47-48 and pp. 65-67]:

$$\sqrt{n+\frac{1}{4}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{2}},$$

then this result was improved by Chu [3]:

$$\sqrt{n+\frac{1}{4}-\frac{1}{(4n-2)^2}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{16n-4}},$$

and then by Boyd [1] and Slavič [23] as:

$$\sqrt{n+\frac{1}{4}+\frac{1}{32n+32}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{32n-\frac{64n-148}{8n+11}}}.$$

Motivated by these formulas, Mortici [5] proposed the following approximations family:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[2k]{P_k(n)}, \tag{1.1}$$

where $P_k(n)$ is a polynomial of kth order (the notation "$f(n) \approx g(n)$" means that the ratio $f(n)/g(n)$ tends to 1, as n approaches infinity). Mortici calculated in [5] the first approximations as $n \to \infty$:

$$\begin{aligned}
\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} &\approx \sqrt[4]{n^2+\frac{1}{2}n+\frac{1}{8}} \\
\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} &\approx \sqrt[6]{n^3+\frac{3}{4}n^2+\frac{9}{32}n+\frac{5}{128}} \\
\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} &\approx \sqrt[8]{n^4+n^3+\frac{1}{2}n^2+\frac{1}{8}n}.
\end{aligned}$$

In [5, p. 427] it is shown that these approximations are increasingly accurate as the root order grows.

Mortici used an original method, however, this method doesn't allow us to determine the general formula of this approximation.

The aim of this paper is to give a systematically method for obtaining the approximations (1.1) for any order $2k$.

The method we propose is related to the theory of asymptotic series and it is inspired from a recent result of Chen and Lin [2].

2. The theoretical results

The asymptotic theory is a strong tool for improving and obtaining new approximation formulas.

Let $f:(0,\infty)\to\mathbb{R}$ be a function. We say that $\sum\limits_{k=1}^{\infty}\frac{\alpha_k}{x^k}$ is an asymptotic series expansion for $f(x)$ as $x\to\infty$, and denote

$$f(x)\sim\sum_{k=1}^{\infty}\frac{\alpha_k}{x^k}\qquad\text{as}\ \ x\to\infty,$$

if for all $m\in\mathbb{N}^*$

$$f(x)-\sum_{k=1}^{m}\frac{\alpha_k}{x^k}=\mathcal{O}\left(\frac{1}{x^{m+1}}\right)\qquad\text{as}\ \ x\to\infty.$$

For a positive function f we write

$$f(x)\sim\exp\left\{\sum_{k=1}^{m}\frac{\alpha_k}{x^k}\right\}\qquad\text{as}\ \ x\to\infty,$$

if for all $m\in\mathbb{N}^*$

$$\ln f(x)-\sum_{k=1}^{m}\frac{\alpha_k}{x^k}=\mathcal{O}\left(\frac{1}{x^{m+1}}\right)\qquad\text{as}\ \ x\to\infty.$$

Using the idea first presented by Chen and Lin in [2], we give the following theorem:

Theorem 2.1. *If the function f has the asymptotic expansion as $x\to\infty$:*

$$f(x)\sim\exp\left\{\sum_{k=1}^{\infty}\frac{\alpha_k}{x^k}\right\}\quad(x>0)\,,$$

then

$$f(x)\sim\sqrt[r]{1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}}\quad(r,x>0)\,,$$

where

$$b_j=\sum_{k_1+2k_2+\ldots+jk_j=j}\frac{r^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}.$$

Proof. This proof is based on the ideas of Chen and Lin presented in [2]. We have

$$f(x)=\exp\left\{\sum_{k=1}^{m}\frac{\alpha_k}{x^k}+R_m(x)\right\},$$

where

$$R_m(x)=\mathcal{O}\left(\frac{1}{x^{m+1}}\right).$$

Thus

$$\begin{aligned}
[f(x)]^r &= e^{r\cdot R_m(x)}\cdot\exp\left\{\sum_{k=1}^{m}\frac{r\alpha_k}{x^k}\right\}\\
&= e^{r\cdot R_m(x)}\prod_{k=1}^{m}\left\{1+\frac{r\alpha_k}{x^k}+\frac{1}{2!}\cdot\left(\frac{r\alpha_k}{x^k}\right)^2+\ldots\right\}\\
&= e^{r\cdot R_m(x)}\sum_{k_1,k_2,..k_m=0}^{\infty}\frac{1}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\left(\frac{r\alpha_1}{x}\right)^{k_1}\cdot\left(\frac{r\alpha_2}{x^2}\right)^{k_2}\cdot\ldots\cdot\left(\frac{r\alpha_m}{x^m}\right)^{k_m}\\
&= e^{r\cdot R_m(x)}\sum_{k_1,k_2,..k_m=0}^{\infty}\frac{r^{k_1+k_2+\ldots+k_m}}{k_1!\cdot k_2!\cdot\ldots\cdot k_m!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_m^{k_m}\cdot\frac{1}{x^{k_1+2k_2+\ldots+mk_m}}\\
&= 1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}
\end{aligned}$$

where

$$b_j=\sum_{k_1+2k_2+\ldots+jk_j=j}\frac{r^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}$$

The proof is now completed.□

In [23], Slavič gave the following integral representation for every $x>0$:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\sim\sqrt{x}\exp\left\{\sum_{k=1}^{n}\frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}\cdot\int_0^{\infty}\left[\frac{\tanh t}{2t}-\sum_{k=1}^{n}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{k(2k)!}t^{2k-2}\right]e^{-4/x}dt\right\}$$

from which, a more accurate double inequality was established:

$$\sqrt{x}\exp\left(\sum_{k=1}^{2m}\frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}\right)<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x}\exp\left(\sum_{k=1}^{2l-1}\frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}\right)$$

for $x>0$. Here m and l are any natural numbers and B_{2k} for $k\in N$ are Bernoulli numbers defined by the generating function

$$\frac{t}{e^t-1}=\sum_{j=0}^{\infty}\frac{B_j}{j!}t^j\quad\left(|t|<2\pi\right).$$

The following asymptotic formula is presented in [23], as $x\to\infty$:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\sim\sqrt{x}\exp\left\{\sum_{j=1}^{\infty}\frac{\left(1-2^{-2j}\right)B_{2j}}{j(2j-1)x^{2j-1}}\right\},$$

which is equivalent to

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\exp\left\{\sum_{k=1}^{\infty}\frac{\left(2-2^{-k}\right)B_{k+1}}{k(k+1)x^k}\right\} \tag{2.1}$$

(in the last formula, the terms involving $B_{2j+1}=0$ were added, for sake of symmetry).

3. Approximations for $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$

By applying Theorem 1 to the function

$$f(x)=\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)} \quad (x>0)\,. \tag{3.1}$$

with the coefficients of the asymptotic series

$$\alpha_k=\frac{\left(2-2^{-k}\right)B_{k+1}}{k(k+1)}, \tag{3.2}$$

see (2.1), and then replacing r by $2r$, we obtain:

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)}\right)^{2r} \sim 1+\sum_{j=1}^{\infty}\frac{b_j}{x^j},$$

where

$$b_j=\sum_{k_1+2k_2+\ldots+jk_j=j}\frac{(2r)^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}\,. \tag{3.3}$$

Then, we deduce that

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[2r]{x^r+b_1x^{r-1}+\ldots+b_{r-1}x+b_r}$$

where $b_1, b_2, \ldots b_r$ are given in (3.3). Concrete values are presented below:

$$r=1 \Rightarrow b_1=\frac{1}{4} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x+\frac{1}{4}}$$

$$r=2 \Rightarrow b_1=\frac{1}{2}, b_2=\frac{1}{8} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[4]{x^2+\frac{1}{2}x+\frac{1}{8}}$$

$$r=3 \Rightarrow b_1=\frac{3}{4}, b_2=\frac{9}{32}, b_3=\frac{5}{128} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[6]{x^3+\frac{3}{4}x^2+\frac{9}{32}x+\frac{5}{128}}$$

$$r=4 \Rightarrow b_1=1, b_2=\frac{1}{2}, b_3=\frac{1}{8}, b_4=0 \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[8]{x^4+x^3+\frac{1}{2}x^2+\frac{1}{8}x}$$

$$\begin{aligned} r &= 5 \Rightarrow b_1=\frac{5}{4}, b_2=\frac{25}{32}, b_3=\frac{35}{128}, b_4=\frac{75}{2048}, b_5=\frac{3}{8192} \\ &\Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[10]{x^5+\frac{5}{4}x^4+\frac{25}{32}x^3+\frac{35}{128}x^2+\frac{75}{2048}x+\frac{3}{8192}} \end{aligned}$$

4. Approximations for Wallis ratio

Let us now apply once again Theorem 1 to the function f given by (3.1), with α_k given by (3.2). Now we replace r by $-2r$ to obtain:

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)}\right)^{-2r} \sim 1+\sum_{j=1}^{\infty}\frac{b'_j}{x^j}$$

where

$$b'_j = \sum_{k_1+2k_2+\ldots+jk_j=j} \frac{(-2r)^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}. \tag{4.1}$$

Hence

$$\left(\frac{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\right)^{2r} \sim 1+\sum_{j=1}^{\infty}\frac{b'_j}{x^j}$$

where $b_1, b_2, \ldots b_r$ are given in (4.1). Furthermore, we obtain:

$$\left(\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\right)^{2r} \sim \frac{1}{x^r}+\sum_{j=1}^{\infty}\frac{b'_j}{x^{j+r}}$$

and therefore

$$\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \sim \sqrt[2r]{\frac{1}{x^r}+\frac{b'_1}{x^{r+1}}+\frac{b'_2}{x^{r+2}}+\ldots}.$$

Using this result, we obtain the following asymptotic expansion for the Wallis sequence, using the relation:

$$P_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}. \tag{4.2}$$

We get

$$P_n \approx \frac{1}{\sqrt{\pi}}\sqrt[2r]{\frac{1}{n^r}+\frac{b'_1}{n^{r+1}}+\frac{b'_2}{n^{r+2}}+\ldots},$$

which is equivalent to

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[2r]{1+\frac{b'_1}{n}+\frac{b'_2}{n^2}+\ldots}.$$

We present the following particular cases:

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt{1-\frac{1}{4n}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[4]{1-\frac{1}{2n}+\frac{1}{8n^2}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[6]{1-\frac{3}{4n}+\frac{9}{32n^2}-\frac{5}{128n^3}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[8]{1-\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{8n^3}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}} \sqrt[10]{1 - \frac{5}{4n} + \frac{25}{32n^2} - \frac{35}{128n^3} + \frac{75}{2048n^4} - \frac{3}{8192n^5}} := \delta_n. \qquad (4.3)$$

5. Conclusions

Mortici's formula stated in [5]:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[8]{n^4 + n^3 + \frac{1}{2}n^2 + \frac{1}{8}n}$$

can be rewritten using (4.2) in the form

$$P_n \approx \frac{1}{\sqrt{\pi}\sqrt[8]{n^4 + n^3 + \frac{1}{2}n^2 + \frac{1}{8}n}} := \mu_n. \qquad (5.1)$$

Our formula (4.3) gives results of the same order of accuracy with Mortici's formula (5.1). A comparison table is given below:

n	$\lvert P_n - \mu_n \rvert$	$\lvert P_n - \delta_n \rvert$
10	1.4655×10^{-10}	1.8666×10^{-10}
50	4.8252×10^{-15}	4.8432×10^{-15}
100	5.4202×10^{-17}	5.2798×10^{-17}
200	6.0379×10^{-19}	5.7940×10^{-19}
500	1.5718×10^{-21}	1.4948×10^{-21}
1000	1.7395×10^{-23}	1.6493×10^{-23}

The formula (4.3) can be equivalently written in terms of gamma function as follows:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \approx \frac{1}{\sqrt{n}} \sqrt[10]{1 - \frac{5}{4n} + \frac{25}{32n^2} - \frac{35}{128n^3} + \frac{75}{2048n^4} - \frac{3}{8192n^5}}.$$

The associated function satisfies the following properties:

Theorem 5.1. *The function* $\varphi : [2, \infty) \to \mathbb{R}$, *defined by*

$$\begin{aligned} \varphi(x) &= \ln\Gamma\left(x + \frac{1}{2}\right) - \ln\Gamma(x+1) + \frac{1}{2}\ln x \\ &\quad + \frac{1}{10}\ln\left(1 - \frac{5}{4x} + \frac{25}{32x^2} - \frac{35}{128x^3} + \frac{75}{2048x^4} - \frac{3}{8192x^5}\right) \end{aligned}$$

is monotonically increasing and concave.

The proof of this theorem is now classical. The same method was used by Chen and Lin, or Mortici in some of their papers. See, *e.g.*, [2], [6]-[22]. We omit the proof for sake of simplicity.

As

$$\varphi(2) = \ln\frac{3}{4}\sqrt{\frac{\pi}{2}} + \frac{1}{10}\ln\frac{141\,141}{262\,144} := \tau$$

(numerically $\tau = -0.1238\cdots$) and $\lim_{x\to\infty} \varphi(x) = 0$, we deduce that

$$\tau \le \varphi(x) < 0 \qquad (x \in \mathbb{R};\ x \ge 2).$$

By exponentiating this double inequality, we get the following result:

Theorem 5.2. *The following double inequality holds true, for every real number $x \ge 2$:*

$$\begin{aligned} & \frac{\beta}{\sqrt{x}} \sqrt[10]{1 - \frac{5}{4x} + \frac{25}{32x^2} - \frac{35}{128x^3} + \frac{75}{2048x^4} - \frac{3}{8192x^5}} \\ \le \; & \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \\ < \; & \frac{\alpha}{\sqrt{x}} \sqrt[10]{1 - \frac{5}{4x} + \frac{25}{32x^2} - \frac{35}{128x^3} + \frac{75}{2048x^4} - \frac{3}{8192x^5}}. \end{aligned}$$

The constants

$$\begin{aligned} \alpha &= 1.0000 \\ \beta &= e^{\tau} = \frac{3}{4}\sqrt{\frac{\pi}{2}} \cdot \sqrt[10]{\frac{141\,141}{262\,144}} = 0.8835\cdots \end{aligned}$$

are sharp.

Further studies on ratio of gamma functions are highly motivated since a deep knowledge of the quotient $\Gamma(x+a)/\Gamma(x+b)$ $(a, b \in \mathbb{R};\ x \to \infty)$ is required in many problems, such as the theory of Mellin-Barnes integrals, the theory of the generalized weighted mean values, or in the theory of hypergeometric functions.

Acknowledgements. The author would like to thank Prof. Cristinel Mortici for his support and assistance in preparing this paper. He also thanks the reviewer for useful comments and corrections that improved the initial form of this manuscript.

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