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# Partial hyperbolic implicit differential equations with variable times impulses

Said Abbas, Mouffak Benchohra and Boualem Attou Slimani

**Abstract.** In this paper we investigate the existence of solutions for the initial value problems (IVP for short), for a class of functional hyperbolic impulsive implicit differential equations with variable time impulses involving the mixed regularized fractional derivative. Our works will be considered by using Schaefer's fixed point.

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## 1. Introduction

The subject of fractional calculus is as old as the differential calculus since, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), it has been developed up to nowadays. Fractional calculus techniques are widely used in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [18, 23]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [6, 7], Kilbas *et al.* [19], Miller and Ross [22], Samko *et al.* [24], the papers of Abbas and Benchohra [1, 2, 3, 4], Abbas *et al.* [5, 8, 9], Benchohra *et al.* [13], Vityuk and Golushkov [26], and the references therein.

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [12], Lakshmikantham *et al.* [20], the papers of Abbas *et al.* [2, 3, 5] and the references therein. The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [10], Abbas and Benchohra [1, 2], Benchohra *et al.* [12], Frigon and O'Regan [14, 15, 16], Lakshmikantham *et al.* [21], Vityuk [25], Vityuk and Golushkov [26], Vityuk and Mykhailenko [27, 28] and the references cited therein.

In the present article we are concerning by the existence of solutions to fractional order IVP for the system

$$\overline{D}^r_{\theta_k} u(x,y) = f(x,y,u(x,y),\overline{D}^r_{\theta_k} u(x,y)); \text{ if } (x,y) \in J_k, \quad x \neq x_k(u(x,y)),\ k = 0,\dots,m, \tag{1.1}$$

$$u(x^+,y) = I_k(u(x,y)); \text{ if } (x,y) \in J,\ x = x_k(u(x,y)),\ k = 1,\dots,m, \tag{1.2}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1.3}$$

where $a,b > 0$, $J := [0,a] \times [0,b]$, $J_0 = [0,x_1] \times (0,b]$, $J_k := (x_k, x_{k+1}] \times (0,b]$; $k = 1,...,m$, $\theta_k = (x_k, 0)$, $\overline{D}^r_{\theta_k}$ is the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,m$, $\varphi \in AC([0,a])$ and $\psi \in AC([0,b])$.
In the present article, we present an existence result based on Schaefer's fixed point.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from $J$ into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebegue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x,y)\| dydx.$$

**Definition 2.1.** [19, 24] *Let $\alpha \in (0,\infty)$ and $u \in L^1(J)$. The partial Riemann-Liouville integral of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by the expression*

$$I^\alpha_{0,x} u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y) ds, \text{ for a.a. } x \in [0,a] \text{ and all } y \in [0,b],$$

*where $\Gamma(.)$ is the (Euler's) Gamma function defined by*

$$\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t} dt;\ \varsigma > 0.$$

Analogously, we define the integral

$$I_{0,y}^{\alpha}u(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^y (y-s)^{\alpha-1}u(x,s)ds, \text{ for a.a. } x \in [0,a] \text{ and a.a. } y \in [0,b].$$

**Definition 2.2.** [19, 24] *Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Riemann-Liouville fractional derivative of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by*

$$(D_{0,x}^{\alpha}u)(x,y) = \frac{\partial}{\partial x}I_{0,x}^{1-\alpha}u(x,y), \text{ for a.a. } x \in [0,a] \text{ and a.a. } y \in [0,b].$$

Analogously, we define the derivative

$$(D_{0,y}^{\alpha}u)(x,y) = \frac{\partial}{\partial y}I_{0,y}^{1-\alpha}u(x,y), \text{ for a.a. } x \in [0,a] \text{ and a.a. } y \in [0,b].$$

**Definition 2.3.** [19, 24] *Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Caputo fractional derivative of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by the expression*

$${}^cD_{0,x}^{\alpha}u(x,y) = I_{0,x}^{1-\alpha}\frac{\partial}{\partial x}u(x,y), \text{ for a.a. } x \in [0,a] \text{ and a.a. } y \in [0,b].$$

Analogously, we define the derivative

$${}^cD_{0,y}^{\alpha}u(x,y) = I_{0,y}^{1-\alpha}\frac{\partial}{\partial y}u(x,y), \text{ for a.a. } x \in [0,a] \text{ and a.a. } y \in [0,b].$$

**Definition 2.4.** [26] *Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by*

$$(I_{\theta}^{r}u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds.$$

In particular,

$$(I_{\theta}^{\theta}u)(x,y) = u(x,y), \; (I_{\theta}^{\sigma}u)(x,y) = \int_0^x\int_0^y u(s,t)dtds; \text{ for a.a. } (x,y) \in J,$$

where $\sigma = (1,1)$.
For instance, $I_{\theta}^{r}u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_{\theta}^{r}u) \in C(J)$, moreover

$$(I_{\theta}^{r}u)(x,0) = (I_{\theta}^{r}u)(0,y) = 0; \; x \in [0,a], \; y \in [0,b].$$

**Example 2.5.** Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_{\theta}^{r}x^{\lambda}y^{\omega} = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2}, \text{ for a.a. } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

**Definition 2.6.** [26] *Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order $r$ of $u$ is defined by the expression $D_{\theta}^{r}u(x,y) = (D_{xy}^2 I_{\theta}^{1-r}u)(x,y)$ and the Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression ${}^cD_{\theta}^{r}u(x,y) = (I_{\theta}^{1-r}D_{xy}^2u)(x,y)$.*

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(x,y) = ({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y), \text{ for a.a. } (x,y) \in J.$$

**Example 2.7.** Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}, \text{ for a.a. } (x,y) \in J.$$

**Definition 2.8.** [28] *For a function $u : J \to \mathbb{R}^n$, we set*

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

*By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of a function $u(x,y)$, we name the function*

$$\overline{D}_\theta^r u(x,y) = D_\theta^r q(x,y).$$

The function

$$\overline{D}_{0,x}^{r_1} u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

is called the partial $r_1-$order regularized derivative of the function $u : J \to \mathbb{R}^n$ with respect to the variable $x$. Analogously, we define the derivative

$$\overline{D}_{0,y}^{r_2} u(x,y) = D_{0,y}^{r_2}[u(x,y) - u(x,0)].$$

Let $a_1 \in [0,a],\ z^+ = (a_1, 0) \in J,\ J_z = [a_1, a] \times [0,b],\ r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I_{z^+}^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order $r$ of $u$.

**Definition 2.9.** [26]. *For $u \in L^1(J_z, \mathbb{R}^n)$ where $D_{xy}^2 u$ is Lebesque integrable on $J_k;\ k = 0, \ldots, m$, the Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression $({}^cD_{z^+}^r f)(x,y) = (I_{z^+}^{1-r} D_{xy}^2 f)(x,y)$. The Riemann-Liouville fractional-order derivative of order $r$ of $u$ is defined by $(D_{z^+}^r f)(x,y) = (D_{xy}^2 I_{z^+}^{1-r} f)(x,y)$.*

Analogously, we define the derivatives

$$\overline{D}_{z^+}^r u(x,y) = D_{z^+}^r q(x,y),$$

$$\overline{D}_{a_1,x}^{r_1} u(x,y) = D_{a_1,x}^{r_1}[u(x,y) - u(0,y)],$$

and

$$\overline{D}_{a_1,y}^{r_2} u(x,y) = D_{a_1,y}^{r_2}[u(x,y) - u(x,0)].$$

## 3. Existence of solutions

To define the solutions of problems (1.1)-(1.3), we shall consider the space

$$\Omega = \{u : J \to \mathbb{R}^n : \text{ there exist } 0 = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = a$$
$$\text{such that } x_k = x_k(u(x_k, .)), \text{ and } u(x_k^-, .),\ u(x_k^+, .) \text{ exist with}$$
$$u(x_k^-, .) = u(x_k, .);\ k = 0, \dots, m, \text{ and } u \in C(J_k); k = 0, \dots, m\}.$$

This set is a Banach space with the norm

$$\|u\|_\Omega = \max\{\|u_k\|;\ k = 0, ..., m\},$$

where $u_k$ is the restriction of $u$ to $J_k$; $k = 0, ..., m$.

**Definition 3.1.** *A function $u \in \Omega \cap (\cup_{k=0}^m AC(J_k))$ such that $u(x,y)$, $\overline{D}_{x_k,x}^{r_1} u(x,y)$, $\overline{D}_{x_k,y}^{r_2} u(x,y)$, $\overline{D}_{z_k^+}^{r} u(x,y)$; $k = 0, \dots, m$, are continuous for $(x,y) \in J_k$ and $I_{z^+}^{1-r} u(x,y) \in AC(J_k)$ is said to be a solution of (1.1)-(1.3) if $u$ satisfies equation (1.1) on $J_k$, and conditions (1.2), (1.3) are satisfied.*

For the existence of solutions for the problem (1.1)-(1.3) we need the following lemmas

**Lemma 3.2.** [28] *Let a function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem*

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y), \overline{D}_\theta^r u(x,y));\ \text{if } (x,y) \in J := [0,a] \times [0,b], \tag{3.1}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{3.2}$$

*is equivalent to the equation*

$$g(x,y) = f(x,y,\mu(x,y) + I_\theta^r g(x,y), g(x,y)), \tag{3.3}$$

*and if $g \in C(J)$ is the solution of (3.3), then $u(x,y) = \mu(x,y) + I_\theta^r g(x,y)$, where*

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

**Lemma 3.3.** [2] *Let $0 < r_1, r_2 \le 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function $u$ is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds; \\ \quad \text{if } (x,y) \in J_0, \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds; \\ \quad \text{if } (x,y) \in J_k;\ k = 1, \dots, m, \end{cases} \tag{3.4}$$

*if and only if $u$ is a solution of the fractional IVP*

$$^cD_{\theta_k}^r u(x,y) = h(x,y); \quad (x,y) \in J_k;\ k = 0, \dots, m, \tag{3.5}$$

$$u(x_k^+, y) = I_k(u(x_k,y)); \quad y \in [0,b];\ k = 1, \dots, m. \tag{3.6}$$

By Lemmas 3.2 and 3.3, we conclude the following Lemma

**Lemma 3.4.** *Let a function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem (1.1)-(1.3) is equivalent to the equation*

$$g(x,y) = f(x,y,\xi(x,y),g(x,y)), \tag{3.7}$$

*where*

$$\xi(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds; \\ \text{ if } (x,y)\in J_0, \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds; \\ \text{ if } (x,y)\in J_k;\ k=1,\dots,m, \end{cases}$$

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

*And if $g \in C(J)$ is the solution of (3.7), then $u(x,y) = \xi(x,y)$.*

**Theorem 3.5.** (Schaefer) [17] *Let $X$ be a Banach space and $N : X \to X$ completely continuous operator. If the set*

$$E(N) = \{u \in X :\ u = \lambda N(u) \text{ for some } \lambda \in [0,1]\}$$

*is bounded, then $N$ has fixed points.*

Further, we present conditions for the existence of solutions of problem (1.1)-(1.3).

**Theorem 3.6.** *Assume*

$(H_1)$ *The function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous,*

$(H_2)$ *For any $u, v, w, z \in \mathbb{R}^n$ and $(x,y) \in J$, there exist constants $M > 0$ such that*

$$\|f(x,y,u,z)\| \le M(1 + \|u\| + \|z\|),$$

$(H_3)$ *The function $x_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for $k = 1,\dots,m$. Moreover,*

$$0 = x_0(u) < x_1(u) < \ldots < x_m(u) < x_{m+1}(u) = a; \quad \text{for all } u \in \mathbb{R}^n,$$

$(H_4)$ *There exists a constant $M^* > 0$ such that $\|I_k(u)\| \le M^*$; for each $u \in \mathbb{R}^n$ and $k = 1,\dots,m$,*

$(H_5)$ *For all $u \in \mathbb{R}^n$, $x_k(I_k(u)) \le x_k(u) < x_{k+1}(I_k(u))$; for $k = 1,\dots,m$,*

$(H_6)$ *For all $(s,t,u) \in J \times \mathbb{R}^n$ and $k = 1,\dots,m$, we have*

$$x_k'(u)\Big[\varphi'(s) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^s\int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1}g(\theta,\eta)d\eta d\theta\Big] \ne 1,$$

*where*

$$g(x,y) = f(x,y,u(x,y),g(x,y));\ (x,y)\in J.$$

*If*

$$M + \frac{Ma^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.8}$$

*then (1.1)-(1.3) has at least one solution on $J$.*

*Proof.* The proof will be given in several steps.
**Step 1.** Consider the following problem

$$\overline{D}_{\theta}^{r}u(x,y)=f(x,y,u(x,y),\overline{D}_{\theta}^{r}u(x,y));\ \text{if } (x,y)\in J, \tag{3.9}$$

$$u(x,0)=\varphi(x),\ u(0,y)=\psi(y);\ x\in[0,a],\ y\in[0,b],\varphi(0)=\psi(0). \tag{3.10}$$

Transform problem (3.9)-(3.10) into a fixed point problem. Consider the operator $N: C(J)\to C(J)$ defined by

$$(Nu)(x,y)=\mu(x,y)+I_{\theta}^{r}g(x,y),$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,u(x,y),g(x,y)).$$

Lemma 3.2 implies that the fixed points of operator $N$ are solutions of problem (3.9)-(3.10). We shall show that the operator $N$ is continuous and completely continuous.
**Claim 1.** *$N$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n\to u$ in $C(J)$. Let $\eta>0$ be such that $\|u_n\|\le\eta$. Then

$$\begin{aligned}\|(Nu_n)(x,y)-(Nu)(x,y)\| &\le \int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\ &\quad\times\|g_n(s,t)-g(s,t)\|dtds,\end{aligned} \tag{3.11}$$

where $g_n,g\in C(J)$ such that

$$g_n(x,y)=f(x,y,u_n(x,y),g_n(x,y))$$

and

$$g(x,y)=f(x,y,u(x,y),g(x,y)).$$

Since $u_n\to u$ as $n\to\infty$ and $f$ is a continuous function, we get

$$g_n(x,y)\to g(x,y) \text{ as } n\to\infty,\ \text{ for each } (x,y)\in J.$$

Hence, (3.11) gives

$$\|(Nu_n)-(Nu)\|_\infty \le \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g_n-g\|_\infty\to 0 \text{ as } n\to\infty.$$

**Claim 2.** *$N$ maps bounded sets into bounded sets in $C(J)$.*
Indeed, it is enough show that for any $\eta^*>0$, there exists a positive constant $\ell^*>0$ such that, for each

$$u\in B_{\eta^*}=\{u\in C(J):\|u\|_\infty\le\eta^*\},$$

we have $\|N(u)\|_\infty\le\ell^*$. For $(x,y)\in J$, we have

$$\begin{aligned}\|(Nu)(x,y)\| &\le \|\mu(x,y)\|\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|dtds,\end{aligned} \tag{3.12}$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,u(x,y),g(x,y)).$$

By $(H_2)$ we have for each $(x,y)\in J$,

$$\begin{aligned}\|g(x,y)\| &\leq M(1+\|\mu(x,y)+I_\theta^r g(x,y)\|+\|g(x,y)\|)\\ &\leq M\Big(1+\|\mu\|_\infty+\frac{a^{r_1}b^{r_2}\|g(x,y)\|}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big)+M\|g(x,y)\|.\end{aligned}$$

Then, by (3.8) we get

$$\|g(x,y)\| \leq \frac{M(1+\|\mu\|_\infty)}{1-M-\frac{Ma^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}:=\ell.$$

Thus, (3.12) implies that

$$\|N(u)\|_\infty\leq\|\mu\|_\infty+\frac{\ell a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}:=\ell^*.$$

**Claim 3.** *$N$ maps bounded sets into equicontinuous sets in $C(J)$.*
Let $(x_1,y_1),(x_2,y_2)\in J$, $x_1<x_2$, $y_1<y_2$, $B_{\eta^*}$ be a bounded set of $C(J)$ as in Claim 2, and let $u\in B_{\eta^*}$. Then

$$\begin{aligned}&\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|\\ \leq\ &\|\mu(x_2,y_2)-\mu(x_1,y_1)\|\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\\ &-(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\|g(s,t)\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds,\end{aligned}$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,u(x,y),g(x,y)).$$

But $\|g\|_\infty\leq\ell$. Thus

$$\begin{aligned}&\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|\\ \leq\ &\|\mu(x_2,y_2)-\mu(x_1,y_1)\|\\ +\ &\frac{\ell}{\Gamma(1+r_1)\Gamma(1+r_2)}[2y_2^{r_2}(x_2-x_1)^{r_1}+2x_2^{r_1}(y_2-y_1)^{r_2}\\ +\ &x_1^{r_1}y_1^{r_2}-x_2^{r_1}y_2^{r_2}-2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].\end{aligned}$$

As $x_1\to x_2$, $y_1\to y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $N$ is continuous and completely continuous.

**Claim 4.** *A priori bounds.*
Now it remains to show that the set

$$\mathcal{E} = \{u \in C(J) : u = \lambda N(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $(x, y) \in J$, we have

We now show there exists an open set $U \subseteq C(J)$ with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C(J)$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x, y) \in J$, we have

$$u(x,y) = \lambda\mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds.$$

This implies by $(H_2)$ and as in Claim 2 that, for each $(x, y) \in J$, we get $\|u\|_\infty \leq \ell^*$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.5, we deduce that $N$ has a fixed point which is a solution of the problem (3.9)-(3.10). Denote this solution by $u_1$. Define the function

$$r_{k,1}(x,y) = x_k(u_1(x,y)) - x, \quad \text{for } x \geq 0, y \geq 0.$$

Hypothesis $(H_3)$ implies that $r_{k,1}(0, 0) \neq 0$ for $k = 1, \ldots, m$.
If $r_{k,1}(x, y) \neq 0$ on $J$ for $k = 1, \ldots, m$; i.e.,

$$x \neq x_k(u_1(x,y)), \quad \text{on } J \quad \text{for } k = 1, \ldots, m,$$

then $u_1$ is a solution of the problem (1.1)-(1.3).
It remains to consider the case when $r_{1,1}(x, y) = 0$ for some $(x, y) \in J$. Now since $r_{1,1}(0, 0) \neq 0$ and $r_{1,1}$ is continuous, there exists $x_1 > 0, y_1 > 0$ such that $r_{1,1}(x_1, y_1) = 0$, and $r_{1,1}(x, y) \neq 0$, for all $(x, y) \in [0, x_1) \times [0, y_1)$.
Thus by $(H_6)$ we have

$$r_{1,1}(x_1,y_1) = 0 \text{ and } r_{1,1}(x,y) \neq 0, \text{ for all } (x,y) \in [0,x_1) \times [0,y_1] \cup (y_1, b].$$

Suppose that there exist $(\bar{x}, \bar{y}) \in [0, x_1) \times [0, y_1] \cup (y_1, b]$ such that $r_{1,1}(\bar{x}, \bar{y}) = 0$. The function $r_{1,1}$ attains a maximum at some point $(s, t) \in [0, x_1) \times [0, b]$.
Since

$$\overline{D}_\theta^r u_1(x,y) = f(x,y,u_1(x,y), \overline{D}_\theta^r u_1(x,y)), \text{ for } (x,y) \in J,$$

then

$$\frac{\partial u_1(x,y)}{\partial x} \text{ exists, and } \frac{\partial r_{1,1}(s,t)}{\partial x} = x_1'(u_1(s,t))\frac{\partial u_1(s,t)}{\partial x} - 1 = 0.$$

Since

$$\frac{\partial u_1(x,y)}{\partial x} = \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} g_1(s,t)dtds,$$

where

$$g_1(x,y) = f(x,y,u_1(x,y), g_1(x,y)); \ (x,y) \in J.$$

Then

$$x_1'(u_1(s,t))[\varphi'(s) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_0^s \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} g_1(\theta,\eta)d\theta d\eta] = 1,$$

witch contradicts $(H_6)$. From $(H_3)$ we have

$$r_{k,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0,x_1) \times [0,b] \text{ and } k = 1,...m.$$

**Step 2.** In what follows set

$$X_k := [x_k, a] \times [0,b]; \; k = 1,...,m.$$

Consider now the problem

$$\overline{D}^r_{\theta_1} u(x,y) = f(x,y,u(x,y),\overline{D}^r_{\theta_1} u(x,y)); \text{ if } (x,y) \in X_1, \tag{3.13}$$

$$u(x_1^+, y) = I_1(u_1(x_1,y)); \; y \in [0,b]. \tag{3.14}$$

Consider the operator $N_1 : C(X_1) \to C(X_1)$ defined as

$$\begin{aligned}(N_1 u) &= \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds,\end{aligned}$$

where

$$g(x,y) = f(x,y,u(x,y),g(x,y)); \text{ for } (x,y) \in X_1.$$

As in Step 1 we can show that $N_1$ is completely continuous. Now it remains to show that the set $\mathcal{E}^* = \{u \in C(X_1) : u = \lambda N_1(u) \text{ for some } 0 < \lambda < 1\}$ is bounded.
Let $u \in \mathcal{E}^*$, then $u = \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus, from $(H_2)$, $(H_4)$ and the fact that $\|g\|_\infty \leq \ell$ we get for each $(x,y) \in X_1$,

$$\begin{aligned}\|u(x,y)\| &\leq \|\varphi(x)\| + \|I_1(u_1(x_1,y))\| + \|I_1(u_1(x_1,0))\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|g(s,t)\|dtds \\ &\leq \|\varphi\|_\infty + 2M^* + \frac{\ell a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell^{**}.\end{aligned}$$

This shows that the set $\mathcal{E}^*$ is bounded. As a consequence of of Theorem 3.5, we deduce that $N_1$ has a fixed point $u$ which is a solution to problem (3.13)-(3.14). Denote this solution by $u_2$. Define

$$r_{k,2}(x,y) = x_k(u_2(x,y)) - x, \quad \text{for } (x,y) \in X_1.$$

If $r_{k,2}(x,y) \neq 0$ on $(x_1, a] \times [0,b]$ and for all $k = 1,\ldots,m$, then

$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y) \in J_0, \\ u_2(x,y), & \text{if } (x,y) \in [x_1,a] \times [0,b], \end{cases}$$

is a solution of the problem (1.1)-(1.3). It remains to consider the case when $r_{2,2}(x,y) = 0$, for some $(x,y) \in (x_1,a] \times [0,b]$. By $(H_5)$, we have

$$\begin{aligned}r_{2,2}(x_1^+, y_1) &= x_2(u_2(x_1^+, y_1) - x_1 \\ &= x_2(I_1(u_1(x_1,y_1))) - x_1 \\ &> x_1(u_1(x_1,y_1)) - x_1 \\ &= r_{1,1}(x_1,y_1) = 0.\end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $x_2 > x_1,\ y_2 > y_1$ such that $r_{2,2}(x_2, y_2) = 0$, and $r_{2,2}(x, y) \neq 0$ for all $(x, y) \in (x_1, x_2) \times [0, b]$.
It is clear by $(H_3)$ that

$$r_{k,2}(x,y) \neq 0 \quad \text{for all } (x,y) \in (x_1, x_2)] \times [0,b];\ k = 2, \ldots, m.$$

Now suppose that there are $(s,t) \in (x_1, x_2) \times [0,b]$ such that $r_{1,2}(s,t) = 0$. From $(H_5)$ it follows that

$$\begin{aligned} r_{1,2}(x_1^+, y_1) &= x_1(u_2(x_1^+, y_1) - x_1 \\ &= x_1(I_1(u_1(x_1, y_1))) - x_1 \\ &\leq x_1(u_1(x_1, y_1)) - x_1 \\ &= r_{1,1}(x_1, y_1) = 0. \end{aligned}$$

Thus $r_{1,2}$ attains a nonnegative maximum at some point $(s_1, t_1) \in (x_1, a) \times [0, x_2) \cup (x_2, b]$.
Since

$$\overline{D}_{\theta_1}^{r} u_2(x,y) = f(x, y, u_2(x,y), \overline{D}_{\theta_1}^{r} u_2(x,y));\ \text{for } (x,y) \in X_1,$$

then we get

$$\begin{aligned} u_2(x,y) &= \varphi(x) + I_1(u_1(x_1, y)) - I_1(u_1(x_1, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g_2(s,t)\,dt\,ds, \end{aligned}$$

where

$$g_2(x,y) = f(x, y, u_2(x,y), g_2(x,y));\ \text{for } (x,y) \in X_1.$$

Hence

$$\frac{\partial u_2}{\partial x}(x,y) = \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} g_2(s,t)\,dt\,ds,$$

then

$$\frac{\partial r_{1,2}(s_1, t_1)}{\partial x} = x_1'(u_2(s_1, t_1)) \frac{\partial u_2}{\partial x}(s_1, t_1) - 1 = 0.$$

Therefore

$$x_1'(u_2(s_1,t_1))[\varphi'(s_1) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{s_1} \int_0^{t_1} (s_1 - \theta)^{r_1-2}(t_1 - \eta)^{r_2-1} g_2(\theta, \eta)\,d\eta\,d\theta] = 1,$$

which contradicts $(H_6)$.

**Step 3.** We continue this process and take into account that $u_{m+1} := u\Big|_{X_m}$ is a solution to the problem

$$\begin{aligned} &\overline{D}_{\theta_m}^{r} u(x,y) = f(x, y, u(x,y), \overline{D}_{\theta_m}^{r} u(x,y)); \quad \text{a.e. } (x,y) \in (x_m, a] \times [0,b], \\ &u(x_m^+, y) = I_m(u_{m-1}(x_m, y));\ y \in [0,b]. \end{aligned}$$

The solution $u$ of the problem (1.1)-(1.3) is then defined by

$$u(x,y)=\begin{cases}u_1(x,y), & \text{if } (x,y)\in J_0,\\ u_2(x,y), & \text{if } (x,y)\in J_1\\ \cdots & \\ u_{m+1}(x,y), & \text{if } (x,y)\in J_m.\end{cases}$$

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Said Abbas
Laboratory of Mathematics, University of Saïda
P.O. Box 138, 20000, Saïda, Algeria
e-mail: `abbasmsaid@yahoo.fr`

Mouffak Benchohra
Laboratory of Mathematics, University of Sidi Bel-Abbès
P.O. Box 89, 22000, Sidi Bel-Abbès, Algeria
Department of Mathematics, Faculty of Science
King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: `benchohra@univ-sba.dz`

Boualem Attou Slimani
Faculté des Sciences de l'Ingénieur, Université de Tlemcen
B.P. 119, 13000, Tlemcen, Algérie
e-mail: `ba_slimani@yahoo.fr`