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Quantitative uniform approximation by generalized discrete singular operators

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Abstract. Here we study the approximation properties with rates of generalized discrete versions of Picard, Gauss-Weierstrass, and Poisson-Cauchy singular operators. We treat both the unitary and non-unitary cases of the operators above. We establish quantitatively the pointwise and uniform convergences of these operators to the unit operator by involving the uniform higher modulus of smoothness of a uniformly continuous function.

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1. Introduction

This article is motivated mainly by [4], where J. Favard in 1944 introduced the discrete version of Gauss-Weierstrass operator

$$\left(F_n f\right)(x) = \frac{1}{\sqrt{\pi n}} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu}{n}\right) \exp\left(-n\left(\frac{\nu}{n} - x\right)^2\right), \tag{1.1}$$

$n \in \mathbb{N}$, which has the property that $\left(F_n f\right)(x)$ converges to $f(x)$ pointwise for each $x \in \mathbb{R}$, and uniformly on any compact subinterval of $\mathbb{R}$, for each continuous function f $(f \in C(\mathbb{R}))$ that fulfills $|f(t)| \leq Ae^{Bt^2}$, $t \in \mathbb{R}$, where A, B are positive constants.

The well-known Gauss-Weierstrass singular convolution integral operators is

$$\left(W_n f\right)(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(u) \exp\left(-n\left(u - x\right)^2\right) du. \tag{1.2}$$

We are also motivated by [1], [2], and [3] where the authors studied extensively the approximation properties of particular generalized singular integral operators such as Picard, Gauss-Weierstrass, and Poisson-Cauchy as well as the general cases of singular integral operators.These operators are not necessarily positive linear operators.

In this article, we define the discrete versions of the operators mentioned above and we study quantitatively their uniform approximation properties regarding convergence to the unit. We examine thoroughly the unitary and non-unitary cases and their interconnections.

2. Background

In [3] p.271-279, the authors studied smooth general singular integral operators $\Theta_{r,\xi}(f,x)$ defined as follows. Let $\xi > 0$ and let μ_ξ be Borel probability measures on $\mathbb{R}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ they defined

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, ..., r, \\ 1 - \sum\limits_{i=1}^{r} (-1)^{r-i}\binom{r}{i} i^{-n}, & j = 0 \end{cases} \tag{2.1}$$

that is $\sum\limits_{j=0}^{r} \alpha_j = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, they defined

$$\Theta_{r,\xi}(f,x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) d\mu_\xi(t) \tag{2.2}$$

for $x \in \mathbb{R}$.

The operators $\Theta_{r,\xi}$ are not necessarily positive linear operators. Indeed we have:

Let $r = 2$, $n = 3$. Then $\alpha_0 = \frac{23}{8}$, $\alpha_1 = -2$, $\alpha_2 = \frac{1}{8}$. Consider $f(t) = t^2 \geq 0$ and $x = 0$. Then

$$\begin{aligned} \Theta_{2,\xi}\left(t^2; 0\right) &= \int_{-\infty}^{\infty} \left(\sum_{j=0}^{2} \alpha_j j^2 t^2 \right) d\mu_\xi(t) \\ &= -\frac{3}{2} \left(\int_{-\infty}^{\infty} t^2 d\mu_\xi(t) \right) \leq 0, \end{aligned}$$

given that $\int\limits_{-\infty}^{\infty} t^2 d\mu_\xi(t) < \infty$.

Authors assumed that $\Theta_{r,\xi}(f,x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

In [3] p.272, the rth modulus of smoothness finite given as

$$\omega_r(f^{(n)}, h) := \sup_{|t| \leq h} \|\Delta_t^r f^{(n)}(x)\|_{\infty,x} < \infty, \quad h > 0, \tag{2.3}$$

where$\|.\|_{\infty,x}$ is the supremum norm with respect to x, $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, and

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x + jt). \tag{2.4}$$

They introduced also

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}, \tag{2.5}$$

and the even function

$$G_n(t) := \int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, w) dw, \quad n \in \mathbb{N} \tag{2.6}$$

with

$$G_0(t) := \omega_r(f, |t|), \quad t \in \mathbb{R}. \tag{2.7}$$

In [3] p.273, they proved

Theorem 2.1. *The integrals* $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, n$, *are assumed to be finite. Then*

$$\left| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right| \le \int_{-\infty}^{\infty} G_n(t) d\mu_\xi(t). \tag{2.8}$$

Moreover, they showed ([3], p.274)

Corollary 2.2. *Suppose* $\omega_r(f, \xi) < \infty, \xi > 0$. *Then it holds for* $n = 0$ *that*

$$|\Theta_{r,\xi}(f;x) - f(x)| \le \int_{-\infty}^{\infty} \omega_r\left(f, |t|\right) d\mu_\xi\left(t\right). \tag{2.9}$$

Furthermore, by using the inequalities

$$G_n(t) \le \frac{|t|^n}{n!} \omega_r(f^{(n)}, |t|) \tag{2.10}$$

and

$$\omega_r(f, \lambda t) \le (\lambda + 1)^r \omega_r(f, t), \lambda, t > 0, \tag{2.11}$$

they obtained

$$\begin{aligned} K_1 &:= \left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x} \\ &\le \frac{\omega_r(f^{(n)}, \xi)}{n!} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} K_2 &:= \|\Theta_{r,\xi}(f;x) - f(x)\|_\infty \\ &\le \omega_r(f, \xi) \int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t). \end{aligned} \tag{2.13}$$

Additionally, they demonstrated ([3], p.279)

Theorem 2.3. *Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$. Set $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, n$. Suppose also $\omega_r(f^{(n)}, h) < \infty, \forall h > 0$. It is also assumed that*

$$\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty. \tag{2.14}$$

Then

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x} \tag{2.15}$$
$$\leq \frac{\omega_r(f^{(n)}, \xi)}{n!} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t).$$

When $n = 0$, the sum in L.H.S (2.15) collapses.

3. Main Results

Here we study important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ.

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, $0 < \xi \leq 1$, $x \in \mathbb{R}$.

i) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{3.1}$$

we define the generalized discrete Picard operators as

$$P_{r,\xi}^{*}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu)\right) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{3.2}$$

ii) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{3.3}$$

we define the generalized discrete Gauss-Weierstrass operators as

$$W_{r,\xi}^{*}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu)\right) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{3.4}$$

iii) Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}, \tag{3.5}$$

we define the generalized discrete Poisson-Cauchy operators as

$$\Theta_{r,\xi}^{*}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{3.6}$$

Observe that for c constant we have

$$P_{r,\xi}^{*}(c;x) = W_{r,\xi}^{*}(c;x) = \Theta_{r,\xi}^{*}(c;x) = c. \tag{3.7}$$

We assume that the operators $P_{r,\xi}^{*}(f;x)$, $W_{r,\xi}^{*}(f;x)$, and $\Theta_{r,\xi}^{*}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$. This is the case when $\|f\|_{\infty,\mathbb{R}} < \infty$.

iv) Let $f \in C_u(\mathbb{R})$ (uniformly continuous functions) or $f \in C_b(\mathbb{R})$ (continuous and bounded functions). When

$$\mu_\xi(\nu) := \mu_{\xi,P}(\nu) := \frac{e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}, \tag{3.8}$$

we define the generalized discrete non-unitary Picard operators as

$$P_{r,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{3.9}$$

Here $\mu_{\xi,P}(\nu)$ has mass

$$m_{\xi,P} := \frac{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{3.10}$$

We observe that

$$\frac{\mu_{\xi,P}(\nu)}{m_{\xi,P}} = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{3.11}$$

which is the probability measure (3.1) defining the operators $P_{r,\xi}^{*}$.

v) Let $f \in C_u(\mathbb{R})$ or $f \in C_b(\mathbb{R})$.When

$$\mu_\xi(\nu) := \mu_{\xi,W}(\nu) := \frac{e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}, \tag{3.12}$$

with $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $\operatorname{erf}(\infty) = 1$, we define the generalized discrete non-unitary Gauss-Weierstrass operators as

$$W_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{3.13}$$

Here $\mu_{\xi,W}(\nu)$ has mass

$$m_{\xi,W} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{3.14}$$

We observe that

$$\frac{\mu_{\xi,W}(\nu)}{m_{\xi,W}} = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{3.15}$$

which is the probability measure (3.3) defining the operators $W^*_{r,\xi}$.

Clearly, here $P_{r,\xi}(f;x)$, $W_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$.

We present our first result.

Proposition 3.1. *Let $n \in \mathbb{N}$. Then, there exists $K_1 > 0$ such that*

$$\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \leq K_1 < \infty \tag{3.16}$$

for all $\xi \in (0,1]$.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} > 1,$$

then

$$\frac{1}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} < 1.$$

Therefore, we obtain

$$\begin{aligned} & \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \\ & < \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}} \\ & := R_1. \end{aligned} \tag{3.17}$$

We notice that

$$\begin{aligned} R_1 &= 2\sum_{\nu=1}^{\infty} \nu^n \left(1+\frac{\nu}{\xi}\right)^r e^{\frac{-\nu}{\xi}} \\ &= 2\sum_{\nu=1}^{\infty} \left(\nu^n e^{\frac{-\nu}{2\xi}}\right) \left(\left(1+\frac{\nu}{\xi}\right)^r e^{\frac{-\nu}{2\xi}}\right). \end{aligned} \tag{3.18}$$

Since we have $\frac{\nu}{\xi} \geq 1$ for $\nu \geq 1$, we get

$$\left(1+\frac{\nu}{\xi}\right)^r e^{\frac{-\nu}{2\xi}} \leq \frac{2^r \nu^r}{\xi^r e^{\frac{\nu}{2\xi}}} = \frac{2^r z^r}{e^{\frac{z}{2}}} \tag{3.19}$$

where $z := \frac{\nu}{\xi}$. Additionally, since

$$e^{\frac{z}{2}} = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \geq \frac{z^r}{2^r r!}, \tag{3.20}$$

we obtain

$$\frac{z^r}{e^{\frac{z}{2}}} \leq 2^r r!. \tag{3.21}$$

Hence, by (3.18), (3.19), and (3.21), we have

$$\begin{aligned} R_1 &\leq 2^{2r+1} r! \sum_{\nu=1}^{\infty} \nu^n e^{\frac{-\nu}{2\xi}} \\ &\leq 2^{2r+1} r! \sum_{\nu=1}^{\infty} \nu^n e^{\frac{-\nu}{2}}. \end{aligned} \tag{3.22}$$

Now, we define the function $f(\nu) = \nu^n e^{\frac{-\nu}{2}}$ for $\nu \geq 1$. Then, we have

$$f'(\nu) = \nu^{n-1} e^{\frac{-\nu}{2}} \left(n - \frac{\nu}{2}\right).$$

Thus, $f(\nu)$ is positive, continuous, and decreasing for $\nu > 2n$. Hence, by shifted triple inequality similar to [5], we get

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \nu^n e^{\frac{-\nu}{2}} \qquad (3.23)\\
&= \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} + \sum_{\nu=2n+1}^{\infty} \nu^n e^{\frac{-\nu}{2}} \\
&\leq \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} + \int_{2n+1}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu + f(2n+1) \\
&\leq \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} + \int_{0}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu + (2n+1)^n e^{-\frac{(2n+1)}{2}} \\
&= \lambda_n + (2n+1)^n e^{-\frac{(2n+1)}{2}} + \int_{0}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu,
\end{aligned}$$

where

$$\lambda_n := \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} < \infty \qquad (3.24)$$

for all $\xi \in (0,1]$. Furthermore, by the integral calculation in [3], p.86, we obtain

$$\int_{0}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu = n!2^{n+1}. \qquad (3.25)$$

Thus, by (3.22), (3.23), and (3.25), we get

$$\begin{aligned}
R_1 &\leq 2^{2r+1} r! \left(\lambda_n + (2n+1)^n e^{-\frac{(2n+1)}{2}} + n!2^{n+1} \right) \qquad (3.26)\\
&< \infty
\end{aligned}$$

for all $\xi \in (0,1]$. Let $K_1 := 2^{2r+1} r! \left(\lambda_n + (2n+1)^n e^{-\frac{(2n+1)}{2}} + n!2^{n+1} \right)$. Then, by (3.17) and (3.26), the proof is done. □

Theorem 3.2. *The sums*

$$c_{k,\xi}^{*} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \ k = 1, ..., n, \qquad (3.27)$$

are finite for all $\xi \in (0,1]$. *Moreover,*

$$\left| P_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^{*} \right| \leq \frac{\sum_{\nu=-\infty}^{\infty} G_n(\nu) e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \qquad (3.28)$$

Clearly the operators $P_{r,\xi}^{*}$ are not necessarily positive operators.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} \nu^{k} e^{\frac{-|\nu|}{\xi}} = \begin{cases} 0, & k \text{ is odd} \\ 2\sum_{\nu=1}^{\infty} \nu^{k} e^{\frac{-\nu}{\xi}}, & k \text{ is even} \end{cases}. \tag{3.29}$$

Assume that k is even. Then, since

$$|\nu|^{k} \leq |\nu|^{n}$$

and

$$1+\frac{|\nu|}{\xi} > 1,$$

we obtain

$$\begin{aligned} & \sum_{\nu=-\infty}^{\infty} \nu^{k} e^{\frac{-|\nu|}{\xi}} \\ = & \sum_{\nu=-\infty}^{\infty} |\nu|^{k} e^{\frac{-|\nu|}{\xi}} \\ \leq & \sum_{\nu=-\infty}^{\infty} |\nu|^{n} \left(1+\frac{|\nu|}{\xi}\right)^{r} e^{\frac{-|\nu|}{\xi}}. \end{aligned} \tag{3.30}$$

Thus, by (3.30) and *Proposition 3.1*, we have

$$c_{k,\xi}^{*} \leq \frac{\sum_{\nu=-\infty}^{\infty} |\nu|^{n} \left(1+\frac{|\nu|}{\xi}\right)^{r} e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} < \infty$$

for all $\xi \in (0,1]$. Therefore, by *Theorem 2.1*, we derive (3.28). □

For $n=0$, we have the following result

Corollary 3.3. *Let $f \in C_{u}(\mathbb{R})$. Then*

$$\left|P_{r,\xi}^{*}(f;x) - f(x)\right| \leq \frac{\sum_{\nu=-\infty}^{\infty} \omega_{r}(f,|\nu|) e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{3.31}$$

Proof. By *Corollary 2.2.* □

Remark 3.4. Inequalities (3.28) and (3.31) give us the uniform estimates

$$\left\| P_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_{k} c_{k,\xi}^{*} \right\|_{\infty,x} \leq \frac{\sum_{\nu=-\infty}^{\infty} G_{n}(\nu) e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \tag{3.32}$$

and

$$\left\|P_{r,\xi}^{*}\left(f;x\right)-f(x)\right\|_{\infty,x}\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}\omega_{r}(f,|\nu|)e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}\tag{3.33}$$

for $n=0$.

Remark 3.5. By (2.12) and (2.13), we obtain

$$\begin{aligned}K_{1}^{*}&:=\left\|P_{r,\xi}^{*}\left(f;x\right)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}c_{k,\xi}^{*}\right\|_{\infty,x}\\&\leq\frac{\omega_{r}(f^{(n)},\xi)}{n!}\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}\right),\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}K_{2}^{*}&:=\left\|P_{r,\xi}^{*}\left(f;x\right)-f(x)\right\|_{\infty,x}\\&\leq\omega_{r}(f,\xi)\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}\right).\end{aligned}\tag{3.35}$$

Hence, by *Proposition 3.1*, for $f^{(n)}\in C_{u}(\mathbb{R})$, we have $K_{1}^{*}\to0$ as $\xi\to0^{+}$ and since

$$\frac{\sum\limits_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}},$$

by *Proposition 3.1*, for $f\in C_{u}(\mathbb{R})$, we get $K_{2}^{*}\to0$ as $\xi\to0^{+}$.

Based on *Remark 3.5, we have*

Theorem 3.6. *Let* $f\in C^{n}(\mathbb{R})$ *with* $f^{(n)}\in C_{u}(\mathbb{R})$, $n\in\mathbb{N}$. *Then*

$$\begin{aligned}&\left\|P_{r,\xi}^{*}\left(f;x\right)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}c_{k,\xi}^{*}\right\|_{\infty,x}\\&\leq\frac{\omega_{r}(f^{(n)},\xi)}{n!}\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}\right).\end{aligned}\tag{3.36}$$

Proof. By *Proposition 3.1* and *Remark 3.5.* □

Next, we present our results for generalized discrete Gauss-Weierstrass operators.

Proposition 3.7. *Let $n \in \mathbb{N}$. Then, there exists $K_2 > 0$ such that*

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \leq K_2 < \infty \tag{3.37}$$

for all $\xi \in (0,1]$.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} > 1.$$

Thus

$$\frac{1}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} < 1.$$

Therefore, we have

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} < \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}. \tag{3.38}$$

On the other hand, since

$$\frac{\nu^2}{\xi} \geq \frac{|\nu|}{\xi},$$

we have

$$e^{\frac{-\nu^2}{\xi}} \leq e^{\frac{-|\nu|}{\xi}}. \tag{3.39}$$

Therefore, by (3.26), (3.38), and (3.39), we have

$$\begin{aligned} & \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \\ < & \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}} \\ = & R_1 < \infty \end{aligned} \tag{3.40}$$

for all $\xi \in (0,1]$. ∎

Theorem 3.8. *The sums*

$$p^*_{k,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \ \ k = 1, ..., n, \tag{3.41}$$

are finite. Furthermore,

$$\left| W_{r,\xi}^{*}\left(f;x\right) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^{*} \right| \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{3.42}$$

Clearly the operators $W_{r,\xi}^{*}$ are not necessarily positive operators.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}} = \begin{cases} 0, & k \text{ is odd} \\ 2\sum\limits_{\nu=1}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}}, & k \text{ is even} \end{cases}. \tag{3.43}$$

Assume that k is even. Then, since

$$|\nu|^k \leq |\nu|^n$$

and

$$1 + \frac{|\nu|}{\xi} > 1,$$

we obtain

$$\begin{aligned} &\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}} \qquad (3.44) \\ &= \sum_{\nu=-\infty}^{\infty} |\nu|^k e^{\frac{-\nu^2}{\xi}} \\ &\leq \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}. \end{aligned}$$

Thus, by (3.44) and *Proposition 3.7*, we have

$$p_{k,\xi}^{*} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} < \infty$$

for all $\xi \in (0,1]$. Therefore, by *Theorem 2.1*, we derive (3.42). □

For $n = 0$, we have the following result.

Corollary 3.9. *Suppose $f \in C_u(\mathbb{R})$. Then*

$$\left| W_{r,\xi}^{*}\left(f;x\right) - f(x) \right| \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f, |\nu|) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{3.45}$$

Proof. By *Corollary 2.2*. □

Remark 3.10. Inequalities (3.42) and (3.45) give us the uniform estimates

$$\left\| W_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^{*} \right\|_{\infty,x} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \tag{3.46}$$

and

$$\left\| W_{r,\xi}^{*}(f;x) - f(x) \right\|_{\infty,x} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{3.47}$$

Remark 3.11. By (2.12) and (2.13), we obtain

$$\begin{aligned} M_1^{*} &:= \left\| W_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^{*} \right\|_{\infty,x} \\ &\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right), \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} M_2^{*} &:= \left\| W_{r,\xi}^{*}(f;x) - f(x) \right\|_{\infty,x} \\ &\leq \omega_r(f,\xi) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right). \end{aligned} \tag{3.49}$$

Hence, by *Proposition 3.7*, for $f^{(n)} \in C_u(\mathbb{R})$, we have $M_1^{*} \to 0$ as $\xi \to 0^{+}$ and since

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}},$$

by *Proposition 3.7*, for $f \in C_u(\mathbb{R})$, we get $M_2^{*} \to 0$ as $\xi \to 0^{+}$.

By previous *Remark 3.11*, we have

Theorem 3.12. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$. Then*

$$\left\| W_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^{*} \right\|_{\infty,x} \tag{3.50}$$
$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right).$$

Proof. By *Proposition 3.7* and *Remark 3.11.* □

Now, we present our results for generalized discrete Poisson-Cauchy operators.

Proposition 3.13. *Let $n \in \mathbb{N}$, $\beta > \frac{n+r+1}{2\alpha}$, and $\alpha \in \mathbb{N}$. Then, there exists $K_3 > 0$ such that*

$$\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq K_3 < \infty \tag{3.51}$$

for all $\xi \in (0,1]$.

Proof. We have

$$\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \tag{3.52}$$
$$= \xi^{-2\alpha\beta} + 2\sum_{\nu=1}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}$$
$$\geq \xi^{-2\alpha\beta}.$$

Therefore

$$\frac{1}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq \xi^{2\alpha\beta}. \tag{3.53}$$

Hence, we get

$$\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{3.54}$$
$$\leq \xi^{2\alpha\beta} \left[\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \right]$$
$$= 2\sum_{\nu=1}^{\infty} \nu^n \left(\xi^{\frac{2\alpha\beta}{r}} + \nu\xi^{\frac{2\alpha\beta}{r}-1}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}.$$

We notice that

$$\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}\leq \nu^{-2\alpha\beta}. \tag{3.55}$$

Thus, by (3.54) and (3.55), we obtain

$$\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{3.56}$$

$$\leq 2\sum_{\nu=1}^{\infty}\nu^{n-2\alpha\beta}\left(\xi^{\frac{2\alpha\beta}{r}}+\nu\xi^{\frac{2\alpha\beta}{r}-1}\right)^r\leq 2\sum_{\nu=1}^{\infty}\nu^{n-2\alpha\beta}\left(1+\nu\right)^r$$

$$\leq 2\sum_{\nu=1}^{\infty}\frac{2^r\nu^r}{\nu^{2\alpha\beta-n}}\leq 2^{r+1}\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}\right)^{2\alpha\beta-n-r}<\infty$$

for all $\xi\in(0,1]$. □

Theorem 3.14. *The sums*

$$q_{k,\xi}^{*}:=\frac{\sum\limits_{\nu=-\infty}^{\infty}\nu^k\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},\ k=1,...,n, \tag{3.57}$$

are finite where $\beta>\frac{n+r+1}{2\alpha}$ *and* $\alpha\in\mathbb{N}$. *Moreover,*

$$\left|\Theta_{r,\xi}^{*}\left(f;x\right)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right|\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}G_n(\nu)\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{3.58}$$

Clearly the operators $\Theta_{r,\xi}^{*}$ *are not necessarily positive operators.*

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty}\nu^k\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}=\begin{cases}0, & k\text{ is odd}\\ 2\sum\limits_{\nu=1}^{\infty}\nu^k\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}, & k\text{ is even}\end{cases}. \tag{3.59}$$

Assume that k is even. Then, since

$$|\nu|^k\leq|\nu|^n\quad\text{and}\quad 1+\frac{|\nu|}{\xi}>1,$$

we obtain

$$\sum_{\nu=-\infty}^{\infty}\nu^k\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}=\sum_{\nu=-\infty}^{\infty}|\nu|^k\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \tag{3.60}$$

$$\leq\sum_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}.$$

Thus, by (3.60) and *Proposition 3.13*, we have

$$q_{k,\xi}^{*} \leq \frac{\sum_{\nu=-\infty}^{\infty} |\nu|^{n} \left(1+\frac{|\nu|}{\xi}\right)^{r} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} < \infty$$

for all $\xi \in (0,1]$. Therefore, by *Theorem 2.1*, we derive (3.58). □

For $n=0$, we have following result.

Corollary 3.15. *Suppose* $f \in C_u(\mathbb{R})$. *Then*

$$\left|\Theta_{r,\xi}^{*}(f;x)-f(x)\right| \leq \frac{\sum_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{3.61}$$

Proof. By *Corollary 2.2*. □

Remark 3.16. Inequalities (3.58) and (3.61) give us the uniform estimates

$$\left\|\Theta_{r,\xi}^{*}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right\|_{\infty,x} \leq \frac{\sum_{\nu=-\infty}^{\infty} G_n(\nu) \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{3.62}$$

and

$$\left\|\Theta_{r,\xi}^{*}(f;x)-f(x)\right\|_{\infty,x} \leq \frac{\sum_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{3.63}$$

Remark 3.17. By (2.12) and (2.13), we obtain

$$\begin{aligned} F_1^{*} &:= \left\|\Theta_{r,\xi}^{*}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right\|_{\infty,x} \qquad (3.64)\\ &\leq \frac{\omega_r(f^{(n)},\xi)}{n!}\left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^{n} \left(1+\frac{|\nu|}{\xi}\right)^{r} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}\right), \end{aligned}$$

and

$$\begin{aligned} F_2^{*} &:= \left\|\Theta_{r,\xi}^{*}(f;x)-f(x)\right\|_{\infty,x} \qquad (3.65)\\ &\leq \omega_r(f,\xi)\left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{r} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}\right). \end{aligned}$$

Hence, by *Proposition 3.13*, for $f^{(n)} \in C_u(\mathbb{R})$, we have $F_1^* \to 0$ as $\xi \to 0^+$ and since

$$\frac{\sum\limits_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^r\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},$$

by *Proposition 3.13*, for $f \in C_u(\mathbb{R})$, we get $F_2^* \to 0$ as $\xi \to 0^+$.

As a conclusion, we state

Theorem 3.18. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, and $\beta > \frac{n+r+1}{2\alpha}$. Then, we have*

$$\left\|\Theta_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^*\right\|_{\infty,x} \tag{3.66}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!}\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}\right).$$

Proof. By *Proposition 3.13* and *Remark 3.17.* □

Remark 3.19. Let μ be a positive finite Borel measure on $\mathbb{R}$ with mass m, i.e. $\mu(\mathbb{R}) = m$. And let $f, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions, $x \in \mathbb{R}$. We observe that

$$\int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x) \tag{3.67}$$

$$= \int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x) - mf(x) + mf(x)$$

$$= \int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - mf(x) + f(x)(m-1).$$

Hence, it holds

$$\left|\int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x)\right| \tag{3.68}$$

$$\leq \left|\int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - mf(x)\right| + |f(x)|\,|m-1|$$

$$\leq m\left|\int_{\mathbb{R}} g_1 \frac{d\mu}{m} + \int_{\mathbb{R}} g_2 \frac{d\mu}{m} - f(x)\right| + |f(x)|\,|m-1|.$$

That is

$$\left|\int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x)\right| \tag{3.69}$$

$$\leq m\left|\int_{\mathbb{R}} g_1 \frac{d\mu}{m} + \int_{\mathbb{R}} g_2 \frac{d\mu}{m} - f(x)\right| + |f(x)|\,|m-1|,$$

where now $\frac{\mu}{m}$ is a probability measure on $\mathbb{R}$.

We prove that $m_{\xi,P} \to 1$ and $m_{\xi,W} \to 1$ as $\xi \to 0^+$. We observe that the function $g(\nu) = e^{-\frac{v}{\xi}}$ is positive, continuous,and decreasing for $\nu \geq 1$. Thus, by [5], we have

$$\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu \leq \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}} \leq e^{-\frac{1}{\xi}} + \int_1^\infty e^{-\frac{\nu}{\xi}} d\nu. \tag{3.70}$$

Thus,

$$1 + 2\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu \leq \sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}} \leq 1 + 2e^{-\frac{1}{\xi}} + 2\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu. \tag{3.71}$$

Since $\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu = \xi e^{-\frac{1}{\xi}}$, we obtain

$$1 + 2\xi e^{-\frac{1}{\xi}} \leq \sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}} \leq 1 + 2e^{-\frac{1}{\xi}} + 2\xi e^{-\frac{1}{\xi}}. \tag{3.72}$$

We have $1 + 2\xi e^{-\frac{1}{\xi}} \to 1$ and $1 + 2e^{-\frac{1}{\xi}} + 2\xi e^{-\frac{1}{\xi}} \to 1$ as $\xi \to 0^+$. Therefore,

$$\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}} \to 1 \text{ as } \xi \to 0^+. \tag{3.73}$$

Thus,

$$m_{\xi,P} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \to 1 \text{ as } \xi \to 0^+. \tag{3.74}$$

Now, define the function $h(\nu) = e^{-\frac{\nu^2}{\xi}}$ for $\nu \geq 1$. Observe that $h(\nu)$ is positive, continuous, and decreasing for $\nu \geq 1$. Then, by [5], we have

$$\int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu \leq \sum_{\nu=1}^{\infty} e^{-\frac{\nu^2}{\xi}} \leq e^{-\frac{1}{\xi}} + \int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu. \tag{3.75}$$

Thus,

$$1 + 2\int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu \leq \sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}} \leq 1 + 2e^{-\frac{1}{\xi}} + 2\int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu. \tag{3.76}$$

As in [2], we have

$$2\int_1^{\infty} e^{-\frac{\nu^2}{\xi}}\,d\nu = \sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right). \tag{3.77}$$

Therefore,

$$\begin{aligned} 1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) &\leq \sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}} \\ &\leq 1+2e^{-\frac{1}{\xi}}+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right). \end{aligned} \tag{3.78}$$

We have $1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)\to 1$ and $1+2e^{-\frac{1}{\xi}}+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)\to 1$ as $\xi\to 0^+$. Hence,

$$\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}} \to 1 \text{ as } \xi\to 0^+. \tag{3.79}$$

Thus,

$$m_{\xi,W} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}}{1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)} \to 1 \text{ as } \xi\to 0^+. \tag{3.80}$$

We define the following error quantities:

$$\begin{aligned} E_{0,P}(f,x) &:= P_{r,\xi}(f;x)-f(x) \\ &= \frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+j\nu)\right)e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} - f(x), \end{aligned} \tag{3.81}$$

$$\begin{aligned} E_{0,W}(f,x) &:= W_{r,\xi}(f;x)-f(x) \\ &= \frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+j\nu)\right)e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} - f(x). \end{aligned} \tag{3.82}$$

Furthermore, we define the errors ($n\in\mathbb{N}$):

$$E_{n,P}(f,x) := P_{r,\xi}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k\frac{\sum_{\nu=-\infty}^{\infty}\nu^k e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \tag{3.83}$$

and

$$E_{n,W}(f,x) := W_{r,\xi}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k\frac{\sum_{\nu=-\infty}^{\infty}\nu^k e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{3.84}$$

Next, working as in inequality (3.69) to the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$, we obtain

$$\left|E_{0,P}(f,x)\right| \leq m_{\xi,P}\left|P_{r,\xi}^{*}(f;x)-f(x)\right|+\left|f(x)\right|\left|m_{\xi,P}-1\right| \tag{3.85}$$

and

$$\left|E_{0,W}(f,x)\right| \leq m_{\xi,W}\left|W_{r,\xi}^{*}(f;x)-f(x)\right|+\left|f(x)\right|\left|m_{\xi,W}-1\right|. \tag{3.86}$$

Furthermore, we obtain $(n \in \mathbb{N})$:

$$\begin{aligned} &\left|E_{n,P}(f,x)\right| \\ \leq\ & m_{\xi,P}\left|P_{r,\xi}^{*}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}c_{k,\xi}^{*}\right|+\left|f(x)\right|\left|m_{\xi,P}-1\right| \end{aligned} \tag{3.87}$$

and

$$\begin{aligned} &\left|E_{n,W}(f,x)\right| \\ \leq\ & m_{\xi,W}\left|W_{r,\xi}^{*}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}p_{k,\xi}^{*}\right|+\left|f(x)\right|\left|m_{\xi,W}-1\right|. \end{aligned} \tag{3.88}$$

Based on *Remark 3.19*, we derive

Theorem 3.20. *It holds*

$$\left|E_{n,P}(f,x)\right| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} G_{n}(\nu)e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right)+\left|f(x)\right|\left|m_{\xi,P}-1\right|. \tag{3.89}$$

Clearly, the operators $P_{r,\xi}(f;x)$ are not necessarily positive operators.

Proof. By (3.28) and (3.87). □

For $n=0$,we have the following result

Corollary 3.21. *Let $f \in C_u(\mathbb{R})$. Then*

$$\left|E_{0,P}(f,x)\right| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_{r}(f,|\nu|)e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right)+\left|f(x)\right|\left|m_{\xi,P}-1\right|. \tag{3.90}$$

Proof. By (3.31) and (3.85). □

We have also the following result

Theorem 3.22. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$,and $\|f\|_{\infty,\mathbb{R}} < \infty$. Then*

$$\begin{aligned} &\left\|E_{n,P}(f,x)\right\|_{\infty,x} \\ \leq\ & \frac{\omega_{r}(f^{(n)},\xi)}{n!}\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right)+\|f\|_{\infty,\mathbb{R}}\left|m_{\xi,P}-1\right|. \end{aligned} \tag{3.91}$$

Proof. By (3.36) and (3.87). □

Next, we present our results for $E_{0,W}(f,x)$ and $E_{n,W}(f,x)$.

Theorem 3.23. *It holds*

$$|E_{n,W}(f,x)| \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} G_n(\nu) e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + |f(x)|\,|m_{\xi,W}-1|. \tag{3.92}$$

Clearly, the operators $W_{r,\xi}(f;x)$ are not necessarily positive operators.

Proof. By (3.42) and (3.88). □

For $n=0$, we have following result

Corollary 3.24. *Let $f \in C_u(\mathbb{R})$. Then*

$$|E_{0,W}(f,x)| \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + |f(x)|\,|m_{\xi,W}-1|. \tag{3.93}$$

Proof. By (3.45) and (3.86). □

We have also the following result

Theorem 3.25. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, and $\|f\|_{\infty,\mathbb{R}} < \infty$. Then*

$$\|E_{n,W}(f,x)\|_{\infty,x} \tag{3.94}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + \|f\|_{\infty,\mathbb{R}}\,|m_{\xi,W}-1|.$$

Proof. By (3.50) and (3.88). □

Conclusion. All of our results presented above imply the higher order of approximation with rates of discrete singular linear operators $P^*_{r,\xi}$, $W^*_{r,\xi}$, $\Theta^*_{r,\xi}$, $P_{r,\xi}$, and $W_{r,\xi}$ to the unit operator I, as $\xi \to 0^+$. Our convergences are pointwise and uniform.

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