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Application of the multi-step homotopy analysis method to solve nonlinear differential algebraic equations

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Abstract. In this paper, a differential algebraic equations (DAE's) is studied and its approximate solution is presented using a multi-step homotopy analysis method (MHAM). The method is only a simple modification of the homotopy analysis method (HAM), in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the corresponding systems. The solutions obtained are also presented graphically. Figurative comparisons between the MHAM and the exact solution reveal that this modified method is very effective and convenient.

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1. Introduction

Differential algebraic equations can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical systems, power systems, chemical process simulation, and optimal control. Many important mathematical models can be expressed in terms of DAEs. In resent years, much research has been focused on the numerical solution of systems of DAEs. For the solutions of nonlinear differential equation, some numerical methods have been developed such as Pade approximation method [6, 7], homotopy perturbation method [11, 10], Adomain decomposition method [15, 5, 4] and variation iteration method [12]. Homotopy analysis method was first introduced by Liao [8], who solved linear and nonlinear problems. The new algorithm, MHAM presented in this paper,accelerates the convergence of the series solution over a large region and improve the accuracy of the HAM. The validity of the modified technique is verified through illustrative examples. The paper is organized as follows. In section 2, the proposed method is described. In section 3, the method is applied to our problem and numerical simulations are presented graphicallly. Finally, the conclusions are given in Section 4.

2. Multi-step homotopy analysis method algorithm

Although the MHAM is used to provide approximate solutions for nonlinear problem in terms of convergent series with easily computable components, it has been shown that the approximated solution obtained are not valid for large t for some systems [13, 9, 14, 1, 2]. Therefore we use the MHAM, which is offers accurate solution over a longer time frame compared to the HAM [16, 3, 17]. For this purpose, we consider the following initial value problem for systems of algebraic differential equations

$$u_i^{'}(t) = f_i(t, u_1, ..., u_n, u_1^{'}, ..., u_n^{'}), \quad t \geq 0, \quad i = 1, 2, ..., n-1,$$
$$0 = g(t, u_1, ..., u_n), \tag{2.1}$$

subject to the initial conditions

$$u_i(0) = c_i, \quad i = 1, 2, ..., n, \tag{2.2}$$

where $(f_i(t),\ i = 1, 2, ..., n-1)$ and g are known analytical functions. Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (2.1) and (2.2). Assume that the interval $[0, T]$ is divided into M subintervals $[t_{j-1}, t_j]$, $j = 1, 2, ..., M$ of equal step size $h = \frac{T}{M}$ by using the nodes $t_j = j\ h$. The main ideas of the MHAM are as follows: Apply the HAM to the initial value problem (2.1) and (2.2) over the interval $[t_0, t_1]$, we will obtain the approximate solution $u_{i,1}$, $t \in [t_0, t_1]$, using the intial condition $(u_i(t_0) = c_i,\ i = 1, 2, ..., n)$. For $j \geq 2$ and at each subinterval $[t_{j-1}, t_j]$ we will use the initial condition $u_{i,j}(t_{j-1}) = u_{i,j-1}(t_{j-1})$ and apply the HAM to the initial value problem (2.1) and (2.2) over the interval $[t_{j-1}, t_j]$. The process is repeated and generates a sequence of approximate solutions $u_{i,j}(t)$, $i = 1, 2, ..., n,\ \ j = 1, 2, ..., M$. Now, we can construct the so-called zeroth-order deformation equations of the system (2.1) by

$$(1-q)L[\phi_{i,j}(t;q) - u_{i,j}(t^*)] = qh[\frac{d}{dt}\phi_{i,j}(t;q) - f_i(t, \phi_{1,j}(t;q), \ldots,$$
$$\phi_{n,j}(t;q), \frac{\partial}{\partial t}\phi_{1,j}(t;q), \ldots, \frac{\partial}{\partial t}\phi_{n,j}(t;q))], \quad i = 1, 2, \ldots, n-1, \tag{2.3}$$
$$(1-q)[\phi_{n,j}(t;q) - u_{n,j}(t^*)] = -qh\ g(t, \phi_{1,j}(t;q), \ldots, \phi_{n,j}(t;q)), \quad j = 1, 2, ..., M,$$

where t^* be the initial value for each subintervals $[t_{j-1}, t_j]$, $q \in [0, 1]$ is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter and $\phi_{i,j}(t;q)$ are unknown functions. Obviously, when $q = 0$ and $q = 1$, we have

$$\phi_{i,j}(t;0) = u_{i,j}(t^*), \phi_{i,j}(t;1) = u_{i,j}(t), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, ..., M,$$

respectively. Expanding $\phi_{i,j}(t;q)$, $i = 1, 2, ..., n,\ \ j = 1, 2, ..., M$, in Taylor series with respect to q, we get

$$\phi_{i,j}(t;q) = u_{i,j}(t^*) + \sum_{m=1}^{\infty} u_{i,j,m}(t)q^m, i = 1, 2, \ldots, n, \quad j = 1, 2, ..., M, \tag{2.4}$$

where

$$u_{i,j,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_{i,j}(t;q)}{\partial q^m}|_{q=0}.$$

If the initial guesses $u_{i,j}(t^*)$, the auxiliary linear operator L and the nonzero auxiliary parameter h are properly chosen so that the power series (2.4) converges at $q = 1$, one has

$$u_{i,j}(t) = \phi_{i,j}(t;1) = u_{i,j}(t^*) + \sum_{m=1}^{\infty} u_{i,j,m}(t),$$

For brevity, define the vector

$$\overrightarrow{u}_{i,j,m}(t) = \{u_{i,j,0}(t), u_{i,j,1}(t), \ldots, u_{i,j,m}(t)\},$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equations

$$\begin{aligned} L[u_{i,j,m}(t) - \chi_m u_{i,j,m-1}(t)] &= h\ \Re_{i,j,m}(\overrightarrow{u}_{i,j,m-1}(t)), \\ u_{n,j,m}(t) &= \chi_m u_{n,j,m-1}(t) + h\ \Re_{n,j,m}(\overrightarrow{u}_{n,j,m-1}(t)) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \Re_{i,j,m}(\overrightarrow{u}_{i,j,m-1}(t)) &= u_{i,j,m-1}(t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[f_i(t, \phi_{1,j}(t;q), \ldots, \phi_{n,j}(t;q), \\ &\quad \frac{\partial}{\partial t}\phi_{1,j}(t;q), \ldots, \frac{\partial}{\partial t}\phi_{n,j}(t;q))])]|_{q=0}, \\ \Re_{n,j,m}(\overrightarrow{u}_{n,j,m-1}(t)) &= \frac{-1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[g(t, \phi_{1,j}(t;q), \ldots, \phi_{n,j}(t;q))]|_{q=0}, \\ i &= 1, 2, ..., n-1, \qquad j = 1, 2, \ldots, M, \end{aligned} \tag{2.6}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}.$$

Select the auxiliary linear operator $L = \frac{d}{dt}$, then the mth-order deformation equations (2.5) can be written in the form

$$\begin{aligned} u_{i,j,m}(t) &= \chi_m u_{i,j,m-1}(t) + h \int_{t_{j-1}}^{t} \Re_{i,j,m}(\overrightarrow{u}_{i,j,m-1}(\tau))d\tau, \\ u_{n,j,m}(t) &= \chi_m u_{n,j,m-1}(t) + h\ \Re_{n,j,m}(\overrightarrow{u}_{n,j,m-1}(t)), \\ i &= 1, 2, ..., n-1, \quad j = 1, 2, ..., M, \end{aligned} \tag{2.7}$$

and a power series solution has the form

$$u_{i,j}(t) = \sum_{m=0}^{\infty} u_{i,j,m}(t),\ i = 1, 2, \ldots, n, \quad j = 1, 2, ..., M. \tag{2.8}$$

Finally, the solutions of system (2.1) has the form

$$u_i(t) = \begin{cases} u_{i,1}(t), & t \in [t_0, t_1] \\ u_{i,2}(t), & t \in [t_1, t_2] \\ \vdots & \\ u_{i,M}(t), & t \in [t_{M-1}, t_M] \end{cases}, \qquad i = 1, 2, ..., n.$$

3. Numerical results

In order to assess both the accuracy and the convergence order of the MHAM presented in this paper for system of differential algebraic equations, we have applied it to the following three problems.

Example 3.1. Consider the following system of differential algebraic equations

$$\begin{aligned} u_1'(t) &= tu_2'(t) - u_1(t) + (1+t)u_2(t), \\ u_2(t) &= \sin t, \end{aligned} \tag{3.1}$$

subject to the initial condition

$$u_1(0) = 1, \ u_2(0) = 0. \tag{3.2}$$

The exact solutions of this system are $(u_1(t) = e^{-t} + t\sin t, \ u_2(t) = \sin t)$. In this example, we apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval $[0, 50]$ to subintervals with time step $\triangle t = 0.1$. In general, we do not have these information at our clearance except at the initial point $t^* = t_0 = 0$, but we can obtain these values by assuming that the new initial condition is the solution in the previous interval. (i.e. If we need the solution in interval $[t_{j-1}, t_j]$, then the initial conditions of this interval will be as

$$\begin{aligned} u_{1,1}(t^*) &= 1, \ u_{1,j}(t^*) = u_{1,j-1}(t_{j-1}) = a_j, \\ u_{2,1}(t^*) &= 0, \ u_{2,j}(t^*) = u_{2,j-1}(t_{j-1}) = b_j, \qquad j = 2, 3, ..., M. \end{aligned} \tag{3.3}$$

Where t^* is the initial value for each subinterval $[t_{j-1}, t_j]$ and a_j, b_j are the initial conditions in the subinterval $[t_{j-1}, t_j]$, $j = 1, 2, ..., M$). In view of the algorithm presented in the previous section, we have the mth-order deformation equation (2.7), where

$$\begin{aligned} \Re_{1,j,m}(\overrightarrow{u}_{1,j,m-1}(t)) &= u_{1,j,m-1}'(t) - tu_{2,j,m-1}' + u_{1,j,m-1}(t) - (1+t)u_{2,j,m-1}(t), \\ \Re_{2,j,m}(\overrightarrow{u}_{2,j,m-1}(t)) &= u_{2,j,m-1}(t) - \sin(t)(1-\chi_m), \\ j &= 1, 2, ..., M, \quad m = 1, 2, 3, \ldots, \end{aligned} \tag{3.4}$$

and the series solution for system (3.1) is given by

$$
\begin{aligned}
u_{1,j,1}(t) &= h\ \left((a_j - b_j)(t-t^*) - \frac{b_j}{2}(t-t^*)^2\right),\\
u_{1,j,2}(t) &= h\ \left((a_j - b_j)(t-t^*) - \frac{b_j}{2}(t-t^*)^2\right) + h^2((a_j - 2b_j)(t-t^*)\\
&\quad + \frac{1}{2}(a_j - b_j)(t-t^*)^2 - \frac{b_j}{6}(t-t^*)^3\\
&\quad - \sin(t-t^*) - (t-t^*)(\cos(t-t^*) + \sin(t-t^*))), \qquad (3.5)\\
&\ \ \vdots
\end{aligned}
$$

Then the 10-term of the approximate solutions of system (3.1) are

$$u_{1,j}(t) = a_j + \sum_{m=1}^{9} u_{1,j,m}(t-t^*).$$

Fig. 1 shows the displacement of the MHAM when $h = -1$ and the exact solution of the system (3.1). It can be seen that the results from the MHAM match the results of the exact solution very well, therefore, the proposed method is very efficient and accurate method that can be used to provide analytical solutions for linear systems of differential algebraic equations.

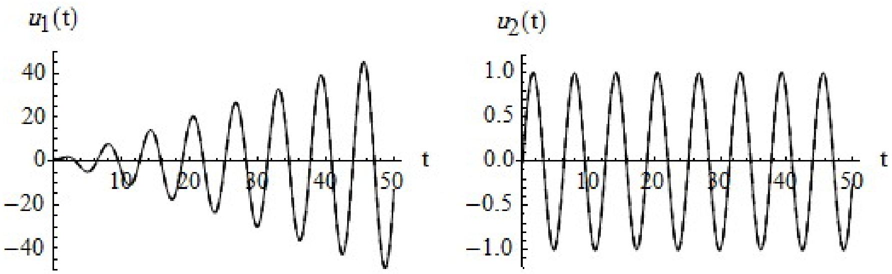

Figure 1. Plots of solution of system (3.1).
Solid line: Exact solution, dashed line: MHAM solution.

Example 3.2. Consider the following nonlinear system of differential algebraic equations

$$
\begin{aligned}
u_1'(t) &= u_1(t)u_2(t) + u_2'(t)(1 - u_2'(t)) - t^2 + 2,\\
u_1^2(t) + u_2^2(t) &+ 2u_1(t)u_2(t) = 4t^2, \qquad (3.6)
\end{aligned}
$$

subject to the initial condition

$$u_1(0) = 0,\ u_2(0) = 0. \qquad (3.7)$$

The exact solutions of this system are $(u_1(t) = t + \sin t,\ \ u_2(t) = t - \sin t)$. Apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval $[0, 50]$ to subintervals with time step $\triangle t = 0.1$. So we start with initial approximation

$$
\begin{aligned}
u_{1,1}(t^*) &= 0,\ u_{1,j}(t^*) = u_{1,j-1}(t_{j-1}) = a_j,\\
u_{2,1}(t^*) &= 0,\ u_{2,j}(t^*) = u_{2,j-1}(t_{j-1}) = b_j, \qquad j = 2, 3, ..., M. \qquad (3.8)
\end{aligned}
$$

In view of the algorithm presented in the previous section, we have the mth-order deformation equation (2.7), where

$$
\begin{aligned}
\Re_{1,j,m}(\overrightarrow{u}_{1,j,m-1}(t)) &= u'_{1,j,m-1}(t) - u'_{2,j,m-1}(t) - \sum_{i=0}^{m-1} u_{1,j,i}(t)u_{2,j,m-i-1}(t) \\
&\quad + \sum_{i=0}^{m-1} u'_{2,j,i}(t)u'_{2,j,m-i-1}(t) + (t^2-2)(1-\chi_m), \\
\Re_{2,j,m}(\overrightarrow{u}_{2,j,m-1}(t)) &= \sum_{i=0}^{m-1} u_{1,j,i}(t)u_{1,j,m-i-1}(t) + \sum_{i=0}^{m-1} u_{2,j,i}(t)u_{2,j,m-i-1}(t) \\
&\quad -2\sum_{i=0}^{m-1} u_{1,j,i}(t)u_{2,j,m-i-1}(t) - 4t^2(1-\chi_m), \qquad (3.9)
\end{aligned}
$$

and the series solution for system (3.6) is given by

$$
\begin{aligned}
u_{1,j,1}(t) &= -h\ \left((a_jb_j+2)(t-t^*) - \frac{1}{3}(t-t^*)^3\right), \\
u_{2,j,1}(t) &= h\ \left((a_j+b_j)^2 - 4(t-t^*)^2\right), \\
u_{1,j,2}(t) &= -h\ \Big((1+h)(a_jb_j+2) + h(a_j^3+2a_j^2b_j+a_jb_j^2))(t-t^*) \\
&\quad -\frac{h}{2}(a_jb_j^2-2b_j-8)(t-t^*)^2 \\
&\quad -\frac{1}{3}(h(4a_j+1)+1)(t-t^*)^3 + \frac{h}{12}b_j(t-t^*)^4\Big), \qquad (3.10) \\
u_{2,j,2}(t) &= h\ \Big((1+2h(a_j+b_j))(a_j+b_j)^2 - 2h(a_jb_j+2)(a_j+b_j)(t-t^*) \\
&\quad -4(1+2h(a_j+b_j))(t-t^*)^2 + \frac{2h}{3}(a_j+b_j)(t-t^*)^3\Big), \\
&\ \ \vdots
\end{aligned}
$$

So, the solution of system (3.6) will be as follows:

$$
\begin{aligned}
u_{1,j}(t) &= a_j + \sum_{m=1}^{9} u_{1,j,m}(t-t^*), \\
u_{2,j}(t) &= b_j + \sum_{m=1}^{9} u_{2,j,m}(t-t^*).
\end{aligned}
$$

Fig. 2 shows the displacement of the MHAM when $h=-1$ and the exact solution of the system (3.6). The results of our computations are in excellent agreement with the results obtained by the exact solution.

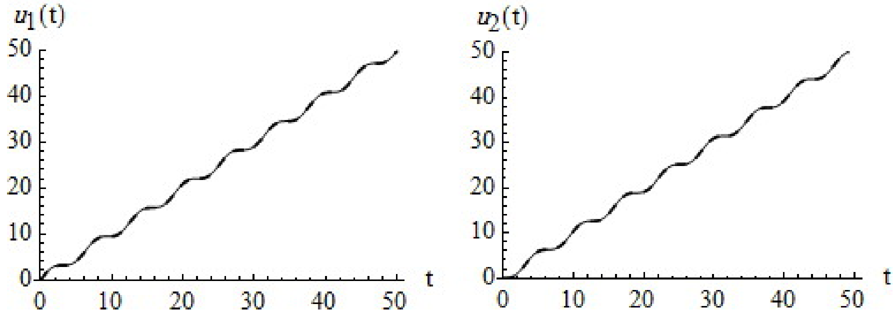

Figure 2. Plots of solution of system (3.6).
Solid line: Exact solution, dashed line: MHAM solution.

Example 3.3. Consider the following nonlinear system of differential algebraic equations

$$\begin{aligned} u_1^{'}(t) &= u_2(t)u_3^{'}(t) - u_2^{'}(t)u_3(t) + u_3^{'}(t) - u_1(t)) + 1 + \sin t, \\ u_2^{'}(t) &= u_3(t)u_1^{'}(t) + u_3(t)u_1(t) + \cos t, \\ & \quad u_1(t)u_2(t)u_3(t) - e^{-t}\sin t\ \cos t, \end{aligned} \tag{3.11}$$

subject to the initial condition

$$u_1(0) = 1,\ u_2(0) = 0,\ u_3(0) = 1 \tag{3.12}$$

The exact solutions of this system are $(u_1(t) = e^{-t},\ u_2(t) = \sin t,\ u_3(t) = \cos t)$. Apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval $[0, 50]$ to subintervals with time step $\triangle t = 0.1$. To solve system (3.11) by means of MHAM, we start with initial approximations

$$\begin{aligned} u_{1,1}(t^*) &= 1,\ u_{1,j}(t^*) = u_{1,j-1}(t_{j-1}) = a_j, \\ u_{2,1}(t^*) &= 0,\ u_{2,j}(t^*) = u_{2,j-1}(t_{j-1}) = b_j, \\ u_{3,1}(t^*) &= 1,\ u_{3,j}(t^*) = u_{3,j-1}(t_{j-1}) = c_j, \quad j = 2, 3, ..., M. \end{aligned} \tag{3.13}$$

In view of the formula (2.7), we can construct the homotopy as follows

$$\begin{aligned} \Re_{1,j,m}(\overrightarrow{u}_{1,j,m-1}(t)) &= u_{1,j,m-1}^{'}(t) - \sum_{i=0}^{m-1} u_{2,j,i}(t)u_{3,j,m-i-1}^{'}(t) \\ &+ \sum_{i=0}^{m-1} u_{2,j,i}^{'}(t)u_{3,j,m-i-1}(t) \\ &+ u_{1,j,m-1}(t) - u_{3,j,m-1}^{'}(t) - (1 + \sin t)(1 - \chi_m), \end{aligned}$$

$$\begin{aligned}
\Re_{2,j,m}(\overrightarrow{u}_{2,j,m-1}(t)) &= u'_{2,j,m-1}(t)\\
&\quad -\sum_{i=0}^{m-1} u_{1,j,i}(t)u_{3,j,m-i-1}(t) - \sum_{i=0}^{m-1} u'_{1,j,i}(t)u_{3,j,m-i-1}(t)\\
&\quad -\cos t\ (1-\chi_m),\\
\Re_{3,j,m}(\overrightarrow{u}_{3,j,m-1}(t)) &= \sum_{i=0}^{m-1} u_{1,j,m-i-1}(t)\sum_{n=0}^{i} u_{2,j,n}(t)u_{3,j,i-n}(t)\\
&\quad -e^{-t}\sin t\ \cos t(1-\chi_m).
\end{aligned}$$

When $h=-1$, the MHAM solution for the system (3.11) in each subinterval $[t_{j-1}, t_j]$ has the form

$$\begin{aligned}
u_{1,j,1}(t) &= 1-(a_j-1)(t-t^*)-\cos(t-t^*),\\
u_{2,j,1}(t) &= a_jc_j(t-t^*)+\sin(t-t^*),\\
u_{3,j,1}(t) &= -a_jb_jc_j+\sin(t-t^*)\cos(t-t^*)\ e^{-(t-t^*)},\\
u_{1,j,2}(t) &= -(1+a_jc_j^2)(t-t^*)-\frac{1}{2}(1-a_j)(t-t^*)^2+(1-c_j)\sin(t-t^*)\\
&\quad +\frac{1}{2}(1+b_j)\sin(2(t-t^*))\ e^{-(t-t^*)},\\
u_{2,j,2}(t) &= c_j+\frac{a_j}{5}+c_j(2-a_j-a_j^2b_j)(t-t^*)+\frac{c_j}{2}(1-a_j)(t-t^*)^2\\
&\quad -c_j(\sin(t-t^*)+\cos(t-t^*))-\frac{a_j}{10}(\sin(2(t-t^*))\\
&\quad +2\cos(2(t-t^*)))\ e^{-(t-t^*)},\\
u_{3,j,2}(t) &= b_jc_j(a_j^2b_j-a_j-1)-c_j(a_j^2c_j+a_jb_j-b_j)(t-t^*)\\
&\quad +c_j(b_j\cos(t-t^*)-a_j\sin(t-t^*))\\
&\quad +(1-a_jb_j)\sin(t-t^*)\cos(t-t^*)\ e^{-(t-t^*)}.\\
&\ \ \vdots
\end{aligned} \tag{3.14}$$

So, the solution of system (3.11) will be as follows:

$$\begin{aligned}
u_{1,j}(t) &= a_j+\sum_{m=1}^{9} u_{1,j,m}(t-t^*),\\
u_{2,j}(t) &= b_j+\sum_{m=1}^{9} u_{2,j,m}(t-t^*),\\
u_{3,j}(t) &= c_j+\sum_{m=1}^{9} u_{3,j,m}(t-t^*).
\end{aligned}$$

Fig. 3 shows the displacement of the MHAM and the exact solution of the system (3.11). From the numerical results in all Figures it is clear that the numerical results obtained using MHAM is in excellent agreement with the exact solutions.

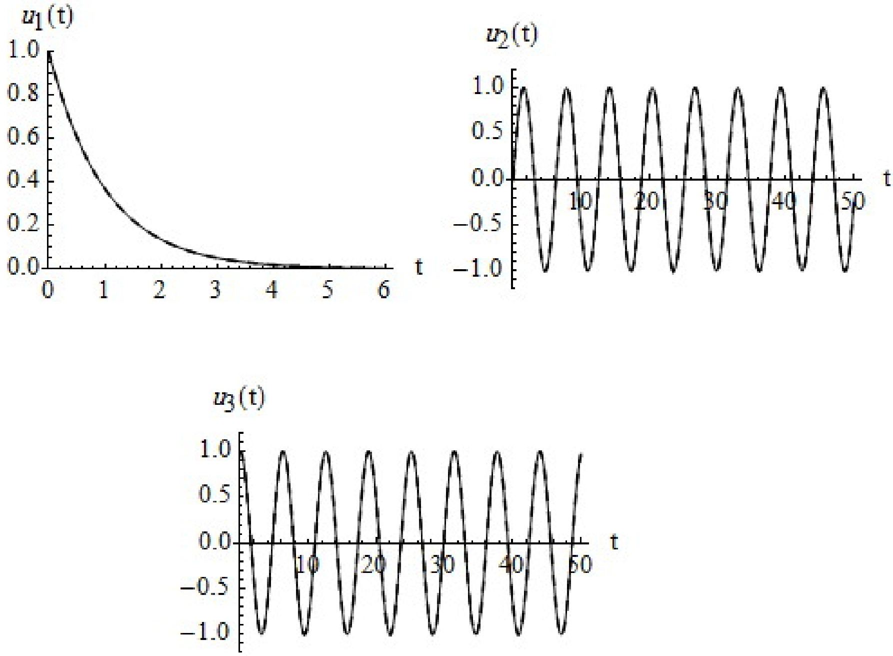

Figure 3. Plots of solution of system (3.11).
Solid line: Exact solution, dashed line: MHAM solution.

4. Conclusions

The purpose of this paper is to construct the multi-step homotopy analysis method to nonlinear systems of differential algebraic equations. The MHAM is that the solution expressed as an infinite series converges very fast to exact solutions. Results have been found very accurate when they are compared with analytical solutions. The approximate solutions obtained by MHAM are highly accurate and valid for a long time. In practice, the utilization of the method is straightforward if some symbolic software as Mathematica is used to implement the calculations. The proposed approach can be further implemented to solve other nonlinear problems.

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