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A note on elliptic problems on the Sierpinski gasket

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Abstract. Using a method that goes back to J. Saint Raymond, we prove the existence of infinitely many weak solutions of certain nonlinear elliptic problems defined on the SG.

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1. Introduction

In the last two decades there has been an increasing interest in studying PDEs on fractals, also motivated and stimulated by the considerable amount of literature devoted to the definition of a Laplace-type operator for functions on fractals. A particular concern has been devoted to PDEs on the Sierpinski gasket. The framework for the study of elliptic equations on the Sierpinski gasket goes back to J. Kigami's pioneering paper [4]. This paper has considerably influenced subsequent papers devoted to PDEs on the Sierpinski gasket. A list of them, including also several recent contributions, may be found in the introduction of [2].

The present paper is devoted to the nonlinear elliptic equation

$$\Delta u(x) + \gamma(x)u(x) = g(x)f(u(x)),$$

defined on the Sierpinski gasket and with zero Dirichlet boundary condition. By imposing that the nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ has an oscillating behavior at ∞, the results of the paper complete those obtained in our previous article [1], where we have studied the same problem, but under the assumption that f oscillates at 0^+. We use, as in [1], a method that goes back to J. Saint Raymond in order to prove that this problem has infinitely many weak solutions. This method has also been used to prove, in

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the context of certain Sobolev spaces, the existence of infinitely many solutions for Dirichlet problems on bounded domains [6], for one-dimensional scalar field equations and systems [3], and for homogeneous Neumann problems [5]. The aim of the present paper is to show that the methods used in [3] can be successfully adapted to the fractal case.

Notations. We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$, by $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidean norm on the spaces $\mathbb{R}^n$, $n \in \mathbb{N}^*$. The spaces $\mathbb{R}^n$ are endowed, throughout the paper, with the Euclidean topology induced by $|\cdot|$.

2. Preliminaries

We briefly recall some notations which will be used in the sequel, and refer to sections 2–4 in [1] for a more detailed presentation of these aspects. Throughout the paper, the letter V stands for the the *Sierpinski gasket* (SG for short) in $\mathbb{R}^{N-1}$, where $N \geq 2$ is a fixed natural number. There are two different approaches that lead to V, starting from given points $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ with $|p_i - p_j| = 1$ for $i \neq j$, and from the similarities $S_i\colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$, defined by

$$S_i(x) = \frac{1}{2}\,x + \frac{1}{2}\,p_i,$$

for $i \in \{1, \dots, N\}$. While in the first approach the set V appears as the unique nonempty compact subset of $\mathbb{R}^{N-1}$ satisfying the equality

$$V = \bigcup_{i=1}^{N} S_i(V),$$

in the second one V is obtained as the closure of the set $V_* := \bigcup_{m\in\mathbb{N}} V_m$, where

$$V_0 := \{p_1, \dots, p_N\} \text{ and } V_m := \bigcup_{i=1}^{N} S_i(V_{m-1}), \text{ for } m \in \mathbb{N}^*.$$

In what follows V is considered to be endowed with the relative topology induced from the topology on $\mathbb{R}^{N-1}$. The set V_0 is called the *intrinsic boundary* of the SG. The natural measure μ associated with V is the normalized restriction of the $\frac{\ln N}{\ln 2}$-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$ to the subsets of V. Thus $\mu(V) = 1$. The Lebesgue spaces $L^p(V, \mu)$, with $p \geq 1$, are equipped with the usual $||\cdot||_p$ norm.

The analog, in the case of the SG, of the Sobolev spaces is the real Hilbert space $H_0^1(V)$, equipped with the inner product $\mathcal{W}\colon H_0^1(V) \times H_0^1(V) \to \mathbb{R}$ which induces the norm $||\cdot||$ (see Section 3 in [1]). The space $H_0^1(V)$ can be compactly embedded in a space of continuous functions. More exactly, if one denotes by $C(V)$ the space of real-valued continuous functions on V, by $C_0(V) := \{u \in C(V) : \ u|_{V_0} = 0\}$, and consider both spaces being endowed with the usual supremum norm $||\cdot||_{\sup}$, then the following Sobolev-type inequality holds for $H_0^1(V)$

$$||u||_{\sup} \leq c||u||, \text{ for every } u \in H_0^1(V), \tag{2.1}$$

where c is a positive constant depending on N. Moreover, the embedding

$$(H_0^1(V), ||\cdot||) \hookrightarrow (C_0(V), ||\cdot||_{\sup}) \tag{2.2}$$

is compact.

As in [1], $\Delta\colon D \to L^2(V,\mu)$ denotes the *weak Laplacian* on V, where D is a certain linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$ (and dense also in $(H_0^1(V), ||\cdot||)$). Thus Δ is bijective, linear, self-adjoint, and satisfies

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v \mathrm{d}\mu, \text{ for every } (u,v) \in D \times H_0^1(V).$$

We recall the following useful property of the space $H_0^1(V)$, stated in Lemma 3.1 of [1].

Lemma 2.1. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with Lipschitz constant $L \geq 0$ and such that $h(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $h \circ u \in H_0^1(V)$ and $||h \circ u|| \leq L \cdot ||u||$.*

3. The main results

Let $\gamma, g \in L^1(V,\mu)$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. We are concerned with the following nonlinear elliptic problem, with zero Dirichlet boundary condition, on the SG

$$(P) \begin{cases} \Delta u(x) + \gamma(x)u(x) = g(x)f(u(x)), \ \forall\, x \in V \setminus V_0, \\ \\ u|_{V_0} = 0. \end{cases}$$

We recall from [1] that a function $u \in H_0^1(V)$ is called a *weak solution of* (P) if

$$\mathcal{W}(u,v) - \int_V \gamma(x)u(x)v(x)\mathrm{d}\mu + \int_V g(x)f(u(x))v(x)\mathrm{d}\mu = 0, \ \forall\, v \in H_0^1(V).$$

Define $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \int_0^t f(\xi)\mathrm{d}\xi. \tag{3.1}$$

We know from Proposition 5.3 in [1] that the functional $T\colon H_0^1(V) \to \mathbb{R}$, given, for every $u \in H_0^1(V)$, by

$$T(u) = \frac{1}{2}||u||^2 - \frac{1}{2}\int_V \gamma(x)u^2(x)\mathrm{d}\mu + \int_V g(x)F(u(x))\mathrm{d}\mu, \tag{3.2}$$

is Fréchet differentiable on $H_0^1(V)$, and that it is an energy functional of problem (P), i.e., $u \in H_0^1(V)$ is a weak solution of (P) if and only if u is a critical point of T.

Remark 3.1. Assume that $\gamma \leq 0$ and $g \leq 0$ a.e. in V. Consider $u \in H_0^1(V)$ and $d, b \in \mathbb{R}$ such that $d \leq u(x) \leq b$ for every $x \in V$. According to the fact that $g \leq 0$ a.e. in V, we then have

$$\int_V g(x)F(u(x))\mathrm{d}\mu \geq \max_{s\in[d,b]} F(s) \cdot \int_V g(x)\mathrm{d}\mu. \tag{3.3}$$

We state, for later use, the following relations about the functional $T\colon H_0^1(V) \to \mathbb{R}$ defined by (3.2): The inequalities (3.3) and $\gamma \le 0$ a.e. in V imply that

$$T(u) \ge \max_{s\in[d,b]} F(s) \cdot \int_V g(x)\mathrm{d}\mu \tag{3.4}$$

and

$$\frac{1}{2}||u||^2 \le T(u) - \max_{s\in[d,b]} F(s) \cdot \int_V g(x)\mathrm{d}\mu. \tag{3.5}$$

We recall the definition of the coercivity of a functional, respectively, the subsequent standard result concerning the existence of minimum points of sequentially weakly lower semicontinuous functionals.

Definition 3.2. Let X be a real normed space and let M be a nonempty subset of X. A functional $L\colon M \to \mathbb{R}$ is said to be *coercive* if, for every sequence (x_n) in M such that $\lim_{n\to\infty} ||x_n|| = \infty$, it follows that $\lim_{n\to\infty} L(x_n) = \infty$.

Proposition 3.3. *Let X be a reflexive real Banach space, M a nonempty sequentially weakly closed subset of X, and $L\colon M \to \mathbb{R}$ a sequentially weakly lower semicontinuous and coercive functional. Then L possesses at least one minimum point.*

We derive now from Proposition 3.3 the following key result for our approach.

Proposition 3.4. *Let $\gamma, g \in L^1(V,\mu)$ be so that $\gamma \le 0$ and $g \le 0$ a.e. in V, let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, and let $a, b, c, d \in \mathbb{R}$ be so that $d < c < 0 < a < b$. Furthermore, assume that the map F, defined by* (3.1)*, satisfies the conditions*

$$F(s) \le F(c), \forall\ s \in [d,c], \tag{3.6}$$

and

$$F(s) \le F(a), \forall\ s \in [a,b]. \tag{3.7}$$

Denoting by

$$M := \{u \in H_0^1(V) \mid d \le u(x) \le b,\ \forall x \in V\},$$

there exists an element $u \in H_0^1(V)$ with the properties:

(i) $T(u) = \inf T(M)$,

(ii) $c \le u(x) \le a$, *for every $x \in V$,*

where the functional $T\colon H_0^1(V) \to \mathbb{R}$ is defined by (3.2)*.*

Proof. Obviously the set M is non-empty (it contains the constant 0 function) and convex. Since the inclusion (2.2) is continuous, M is closed in the norm topology on $H_0^1(V)$. It follows that M is also closed in the weak topology on $H_0^1(V)$, thus M is sequentially weakly closed. It follows from (3.5) that the restriction of T to M is coercive. Proposition 3.3 implies now that there exists $\tilde{u} \in M$ such that $T(\tilde{u}) = \inf T(M)$. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} c, & t < c \\ t, & t \in [c,a] \\ a, & t > a. \end{cases}$$

Note that $h(0) = 0$ and that h is a Lipschitz map with Lipschitz constant $L = 1$. According to Lemma 2.1, the map $u := h \circ \tilde{u}$ belongs to $H_0^1(V)$ and

$$||u|| \leq ||\tilde{u}||. \tag{3.8}$$

Moreover, u belongs to M and obviously satisfies condition (ii) to be proved. We next show that (i) also holds true. For this set

$$V_1 := \{x \in V \mid \tilde{u}(x) < c\}, \quad V_2 := \{x \in V \mid \tilde{u}(x) > a\}.$$

Then

$$u(x) = \begin{cases} c, \ x \in V_1 \\ \tilde{u}(x), \ x \in V \setminus (V_1 \cup V_2) \\ a, \ x \in V_2. \end{cases}$$

It follows that

$$u^2(x) \leq \tilde{u}^2(x), \text{ for every } x \in V. \tag{3.9}$$

Furthermore, if $x \in V_1$, then $\tilde{u}(x) \in [d, c[$, hence, by (3.6), $F(\tilde{u}(x)) \leq F(c) = F(u(x))$. Analogously, if $x \in V_2$, then (3.7) yields $F(\tilde{u}(x)) \leq F(a) = F(u(x))$. Thus

$$F(\tilde{u}(x)) \leq F(u(x)), \text{ for every } x \in V. \tag{3.10}$$

The inequalities (3.8), (3.9) and (3.10) imply, together with the fact that $\gamma \leq 0$ and $g \leq 0$ a.e. in V, that

$$\begin{aligned} T(\tilde{u}) - T(u) &= \frac{1}{2}||\tilde{u}||^2 - \frac{1}{2}||u||^2 - \frac{1}{2}\int_V \gamma(x)(\tilde{u}^2(x) - u^2(x))\mathrm{d}\mu \\ &+ \int_V g(x)(F(\tilde{u}(x)) - F(u(x)))\mathrm{d}\mu \geq 0. \end{aligned}$$

Thus $T(\tilde{u}) \geq T(u)$. Since $T(\tilde{u}) = \inf T(M)$ and since $u \in M$, we conclude that $T(u) = \inf T(M)$, thus (i) is also fulfilled. □

The main result of the paper is contained in the following theorem concerning the existence of multiple weak solutions of problem (P).

Theorem 3.5. *Assume that the following conditions hold:*

(C1) $\gamma \in L^1(V, \mu)$ *and* $\gamma \leq 0$ *a.e. in* V.

(C2) $f\colon \mathbb{R} \to \mathbb{R}$ *is continuous such that*

  (1*) *there exist two sequences* (a_k) *and* (b_k) *in* $]0, \infty[$ *with* $a_k < b_k < b_{k+1}$, $\lim_{k\to\infty} b_k = \infty$ *and such that* $f(s) \leq 0$ *for every* $s \in [a_k, b_k]$,

  (2*) *there exist reals* $d < c < 0$ *with* $f(s) \geq 0$ *for every* $s \in [d, c]$.

(C3) $F\colon \mathbb{R} \to \mathbb{R}$, *defined by* (3.1), *is such that*

  (3*) $-\infty < \liminf_{s\to\infty} \dfrac{F(s)}{s^2}$,

  (4*) $\limsup_{s\to\infty} \dfrac{F(s)}{s^2} = \infty$.

(C4) $g\colon V \to \mathbb{R}$ *is continuous, not identically* 0, *and with* $g \leq 0$.

Then there exists a sequence (u_k) *of pairwise distinct weak solutions of problem* (P) *such that* $\lim_{k\to\infty} \|u_k\| = \infty$.

Remark 3.6. According to Example 4 in [3], the function $f\colon \mathbb{R} \to \mathbb{R}$, defined by $f(s) = s^2 \sin^2 s - 1$, satisfies the conditions (C2) and (C3) of Theorem 3.5.

In what follows we assume that the conditions (C1)–(C4) in the hypotheses of Theorem 3.5 are satisfied. For every $k \in \mathbb{N}$ set now

$$M_k := \{u \in H_0^1(V) \mid d \leq u(x) \leq b_k,\ \forall\ x \in V\}. \tag{3.11}$$

The proof of Theorem 3.5 includes the following main steps contained in the next results:

1. we show that the functional $T\colon H_0^1(V) \to \mathbb{R}$, defined by (3.2), has at least one critical point in each of the sets M_k,
2. since T is an energy functional of Problem (P), each of these critical points is a weak solution of Problem (P),
3. we show that there are infinitely many pairwise distinct such weak solutions.

Lemma 3.7. *For every $k \in \mathbb{N}$, there is an element $u_k \in M_k$ such that the following conditions hold:*

(i) $T(u_k) = \inf T(M_k)$,

(ii) $c \leq u_k(x) \leq a_k$, *for every $x \in V$.*

Proof. Note that, while condition (1^*) in the hypotheses of Theorem 3.5 yields

$$F(s) \leq F(a_k), \forall\ s \in [a_k, b_k],$$

condition (2^*) implies (3.6). Applying Proposition 3.4, we finish the proof. □

Lemma 3.8. *For every $k \in \mathbb{N}$, let $u_k \in M_k$ be a function satisfying the conditions* (i) *and* (ii) *of Lemma* 3.7*. The functional T has then in u_k a local minimum (with respect to the norm topology on $H_0^1(V)$), for every $k \in \mathbb{N}$. In particular, (u_k) is a sequence of weak solutions of problem (P).*

Proof. Fix $k \in \mathbb{N}$. Suppose to the contrary that u_k is not a local minimum of T. This implies the existence of a sequence (w_n) in $H_0^1(V)$ converging to u_k in the norm topology such that

$$T(w_n) < T(u_k), \text{ for every } n \in \mathbb{N}.$$

In particular, $w_n \notin M_k$, for all $n \in \mathbb{N}$. Choose a real number ε such that

$$0 < \varepsilon \leq \frac{1}{2} \min\{b_k - a_k, c - d\}.$$

In view of (2.1) the sequence (w_n) converges to u_k in the supremum norm topology on $C(V)$. Hence there is an index $m \in \mathbb{N}$ such that

$$||w_m - u_k||_{\sup} \leq \varepsilon.$$

For every $x \in V$ we then have, according to condition (ii) of Lemma 3.7,

$$w_m(x) = w_m(x) - u_k(x) + u_k(x) \leq \varepsilon + u_k(x) \leq \frac{b_k - a_k}{2} + a_k < b_k$$

and

$$w_m(x) = w_m(x) - u_k(x) + u_k(x) \geq -\varepsilon + u_k(x) \geq \frac{d - c}{2} + c > d.$$

Thus $w_m \in M_k$, a contradiction. We conclude that T has in u_k a local minimum, so u_k is a critical point of T. The last assertion of the lemma follows now from the fact that T is an energy functional of problem (P). $\square$

Lemma 3.9. *For every $k \in \mathbb{N}$, put $\gamma_k := \inf T(M_k)$. Then $\lim_{k\to\infty} \gamma_k = -\infty$.*

Proof. Observe first that the inclusions $M_k \subseteq M_{k+1}$, for all $k \in \mathbb{N}$, imply that the sequence (γ_k) is decreasing.

Condition (C4) in Theorem 3.5 yields the existence of a nonempty open subset U of $V \setminus V_0$ such that $g|_U < 0$. By the same arguments as those used in the proof of statement (2.1) in [1] we may conclude that there exists a compact set $K \subseteq U$ with $\mu(K) > 0$. Hence we get that

$$\int_K g(x)\mathrm{d}\mu < 0. \tag{3.12}$$

We show next that we can find a function $v \in H_0^1(V)$ such that

$$0 \le v \le 1 \text{ and } v|_K = 1. \tag{3.13}$$

Indeed, by Urysohn's Lemma, there exists a continuous function $\phi\colon V \to [0,1]$ such that $\phi(x) = 0$, for $x \in V_0$, and $\phi(x) = 1$, for $x \in K$. According to Theorem 1.4.4 in [7], there exists a function $u \in H_0^1(V)$ with $||\phi - u||_{\sup} < 1$. In particular, $u(x) \neq 0$ for all $x \in K$. Hence $|u(x)| > 0$ for every $x \in K$. Note that $|u| \in H_0^1(V)$, by Lemma 2.1. Let

$$\xi := \min_{x\in K} |u(x)|.$$

Then $\xi > 0$. Define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(t) = \min\{t, \xi\}$. Since h is a Lipschitz map with $h(0) = 0$, Lemma 2.1 yields that $h \circ |u| \in H_0^1(V)$. We have that $(h \circ |u|)(x) = \xi$ for every $x \in K$. Thus $v := \frac{1}{\xi}(h \circ |u|)$ satisfies (3.13).

By condition (3*) in the requirements of Theorem 3.5, there exist $m \in \mathbb{R}$ and $\delta > 0$ such that

$$m \le \frac{F(s)}{s^2}, \text{ for all } s > \delta.$$

Denote by $\widetilde{m} := \min\{F(s) - ms^2 \mid s \in [0,\delta]\}$. In particular, $\widetilde{m} \le 0$. So we obtain that

$$\widetilde{m} + ms^2 \le F(s), \text{ for all } s \ge 0. \tag{3.14}$$

Condition (4*) in the hypotheses of Theorem 3.5 implies the existence of a sequence (r_n) of positive reals with

$$\lim_{n\to\infty} r_n = \infty \text{ and } \lim_{n\to\infty} \frac{F(r_n)}{r_n^2} = \infty. \tag{3.15}$$

Using (3.2) and (3.13), we compute, for every $n \in \mathbb{N}$,

$$\begin{aligned} T(r_n v) =& \frac{1}{2} r_n^2 ||v||^2 - \frac{r_n^2}{2} \int_V \gamma(x) v^2(x) \mathrm{d}\mu + F(r_n) \int_K g(x)\mathrm{d}\mu \\ &+ \int_{V\setminus K} g(x) F(r_n v(x))\mathrm{d}\mu. \end{aligned}$$

On the other hand, by (3.14) and the fact that $g \leq 0$, we get

$$\int_{V\setminus K} g(x)F(r_n v(x))\mathrm{d}\mu \leq \widetilde{m}\int_{V\setminus K} g(x)\mathrm{d}\mu + mr_n^2\int_{V\setminus K} g(x)v^2(x)\mathrm{d}\mu.$$

Thus

$$\begin{aligned}\frac{T(r_n v)}{r_n^2} \leq & \frac{||v||^2}{2} - \frac{1}{2}\int_V \gamma(x)v^2(x)\mathrm{d}\mu + \frac{F(r_n)}{r_n^2}\int_K g(x)\mathrm{d}\mu \\ & + \frac{\widetilde{m}}{r_n^2}\int_{V\setminus K} g(x)\mathrm{d}\mu + m\int_{V\setminus K} g(x)v^2(x)\mathrm{d}\mu.\end{aligned} \tag{3.16}$$

Involving (3.12) and (3.15), we obtain from (3.16) that $\lim_{n\to\infty}\frac{T(r_n v)}{r_n^2} = -\infty$, so

$$\lim_{n\to\infty} T(r_n v) = -\infty. \tag{3.17}$$

Recall from condition (1*) in the statement of Theorem 3.5 that $\lim_{k\to\infty} b_k = \infty$. Thus we may find a subsequence (b_{k_n}) of the sequence (b_k) such that $r_n \leq b_{k_n}$, for every $n \in \mathbb{N}$. Since $0 \leq v \leq 1$, we get that

$$0 \leq r_n v \leq b_{k_n}, \text{ for all } n \in \mathbb{N}.$$

By (3.11), we hence conclude that $r_n v \in M_{k_n}$, for every $n \in \mathbb{N}$, so

$$\gamma_{k_n} \leq T(r_n v), \text{ for all } n \in \mathbb{N}.$$

In view of (3.17) we thus obtain that $\lim_{n\to\infty}\gamma_{k_n} = -\infty$. Since (γ_k) is decreasing we finally conclude that $\lim_{k\to\infty}\gamma_k = -\infty$. □

Proof of Theorem 3.5 *concluded.* From Lemma 3.8 we know that there is a sequence (u_k) of weak solutions of problem (P) such that $u_k \in M_k$ and $\gamma_k = T(u_k)$, where $\gamma_k = \inf T(M_k)$, for every natural k. Assume, by contradiction, that $\lim_{k\to\infty} ||u_k|| \neq \infty$. Then there exists a bounded subsequence (u_{k_n}) of the sequence (u_k). According to (2.1) and to the fact that $\lim_{k\to\infty} b_k = \infty$, we may find $p \in \mathbb{N}$ such that $u_{k_n} \in M_p$, for every $n \in \mathbb{N}$. This yields that $\gamma_p \leq \gamma_{k_n}$, for every $n \in \mathbb{N}$, contradicting the statement of Lemma 3.9. Thus $\lim_{k\to\infty} ||u_k|| = \infty$. Hence we can find a subsequence (u_{k_j}) of the sequence (u_k) consisting of pairwise distinct elements. □

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