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# The weighted mean operator on $\ell^2$ with weight sequence $w_n = (n+1)^p$ is hyponormal for $p = 2$

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**Abstract.** Posinormality is used to demonstrate that the weighted mean matrix whose weight sequence is the sequence of squares of positive integers is a hyponormal operator on $\ell^2$.

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## 1. Introduction

In this paper, attention will be focused on an example of a weighted mean matrix that does not satisfy the sufficient conditions for hyponormality given in [**4**]. Nor does it satisfy the key lemma used in [5], so a somewhat different approach will be required here. The computations here are much more complex than those in [5], and, because of that, the computer software package SAGE [6] has been used as an aid.

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space $H$, then $A \in B(H)$ is said to be is *posinormal* (see [1], [2]) if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, called the *interrupter*, and $A$ is *hyponormal* if $A^*A - AA^* \geq 0$. Hyponormal operators are necessarily posinormal.

A lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on $\ell^2$, is *factorable* if its entries are of the form

$$m_{ij} = \begin{cases} a_i c_j & if \quad j \leq i \\ 0 & if \quad j > i \end{cases}$$

where $a_i$ depends only on $i$ and $c_j$ depends only on $j$; the matrix $M$ is *terraced* if $c_j = 1$ for all $j$. A *weighted mean matrix* is a lower triangular matrix with entries $w_j/W_i$, where $\{w_j\}$ is a nonnegative sequence with $w_0 > 0$, and $W_i = \sum_{j=0}^{i} w_j$. A weighted mean matrix is factorable, with $a_i = 1/W_i$ and $c_j = w_j$ for all $i$,$j$.

## 2. Main result

Under consideration here will be the weighted mean matrix $M$ associated with the weight sequence $w_n = (n+1)^2$. As was also the case for $w_n = n+1$, this example is not easily seen to be hyponormal directly from the definition and fails to satisfy the sufficent conditions for hyponormality given in [4, Corollary 1], but it survives the necessary condition given in [4, Corollary 2]. Encouraged by the latter, we set out to prove that $M$ is hyponormal. The next theorem will provide us our main tool – an expression for the interrupter $P$ associated with the matrix $M$.

**Theorem 2.1.** *Suppose $M = [a_i c_j]$ is a lower triangular factorable matrix that acts as a bounded operator on $\ell^2$ and that the following conditions are satisfied:*

(a) *both $\{a_n\}$ and $\{a_n/c_n\}$ are positive decreasing sequences that converge to 0, and*
(b) *the matrix $B$ defined by $B = [b_{ij}]$ by*

$$b_{ij} = \begin{cases} c_i(\frac{1}{c_j} - \frac{1}{c_{j+1}}\frac{a_{j+1}}{a_j}) & if \quad i \leq j; \\ -\frac{a_{j+1}}{a_j} & if \quad i = j+1; \\ 0 & if \quad i > j+1. \end{cases}$$

*is a bounded operator on $\ell^2$.*
*Then $M$ is posinormal with interrupter $P = B^*B$. The entries of $P = [p_{ij}]$ are given by*

$$p_{ij} = \begin{cases} \frac{c_j^2 c_{j+1}^2 a_{j+1}^2 + (\sum_{k=0}^{j} c_k^2)(c_{j+1}a_j - c_j a_{j+1})^2}{c_j^2 c_{j+1}^2 a_j^2} & if \quad i = j; \\ \frac{(c_i a_{i+1} - c_{i+1} a_i)[c_j(\sum_{k=0}^{j+1} c_k^2)a_{j+1} - c_{j+1}(\sum_{k=0}^{j} c_k^2)a_j]}{c_i c_{i+1} c_j c_{j+1} a_i a_j} & if \quad i > j; \\ \frac{(c_j a_{j+1} - c_{j+1} a_j)[c_i(\sum_{k=0}^{i+1} c_k^2)a_{i+1} - c_{i+1}(\sum_{k=0}^{i} c_k^2)a_i]}{c_i c_{i+1} c_j c_{j+1} a_i a_j} & if \quad i < j. \end{cases}$$

*Proof.* See [3]. □

We are now ready for the main result. The induction step in the proof below was aided by explicit computations using the computer software package SAGE [6].

**Theorem 2.2.** *The weighted mean matrix $M$ associated with the weight sequence $w_n = (n+1)^2$ is hyponormal.*

*Proof.* One easily verifies that the weighed mean matrix $M$ associated with $w_n = (n+1)^2$ satisfies the hypotheses of Theorem 2.1. For $M$ to be hyponormal, we must have

$$\langle (M^*M - MM^*)f, f\rangle = \langle (M^*M - M^*PM)f, f\rangle = \langle (I-P)Mf, Mf\rangle \geq 0$$

for all $f$ in $\ell^2$. Consequently, we can conclude that $M$ will be hyponormal when $Q :\equiv I - P \geq 0$; we note that the range of M contains all the $e_n$'s from the standard orthonormal basis for $\ell^2$.

Using the given weight sequence, we determine that the entries of $Q = [q_{mn}]$ are given by

$$q_{mn} = \begin{cases} \frac{60n^7+600n^6+2488n^5+5476n^4+6795n^3+4650n^2+1584n+207}{30(n+1)^3(n+2)^3(n+3)(2n+5)} & if \quad m = n; \\ -\frac{1}{30} \cdot \frac{10m^3+52m^2+93m+57}{(m+1)^2(m+2)^2(m+3)(2m+5)} \cdot \frac{(3n^2+7n+3)(2n+3)}{(n+1)(n+2)} & if \quad m > n; \\ -\frac{1}{30} \cdot \frac{10n^3+52n^2+93n+57}{(n+1)^2(n+2)^2(n+3)(2n+5)} \cdot \frac{(3m^2+7m+3)(2m+3)}{(m+1)(m+2)} & if \quad m < n. \end{cases}$$

In order to show that $Q$ is positive, it suffices to show that $Q_N$, the $N^{th}$ finite section of $Q$ (involving rows $m = 0, 1, 2, ..., N$ and columns $n = 0, 1, 2, ..., N$), has positive determinant for each positive integer $N$. For columns $n = 0, 1, ..., N-1$, we multiply the $(n+1)^{st}$ column of $Q_N$ by

$$z_n :\equiv \frac{(n+3)(2n+3)(3n^2+7n+3)}{(n+1)(2n+5)(3n^2+13n+13)}$$

and subtract from the $n^{th}$ column. Call the new matrix $Q'_N$. Then we work with the rows of $Q'_N$. For $m = 0, 1, ..., N-1$, we multiply the $(m+1)^{st}$ row of $Q'_N$ by $z_m$ and subtract from the $m^{th}$ row. This leads to the tridiagonal form

$$Y_N :\equiv \begin{pmatrix} d_0 & s_0 & 0 & \dots & 0 & 0 \\ s_0 & d_1 & s_1 & \dots & 0 & 0 \\ 0 & s_1 & d_2 & \dots & . & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & . & \dots & d_{N-1} & s_{N-1} \\ 0 & 0 & 0 & \dots & s_{N-1} & d_N \end{pmatrix},$$

where

$$d_n = q_{nn} - z_n q_{n,n+1} - z_n(q_{n+1,n} - z_n q_{n+1,n+1}) = q_{nn} - 2z_n q_{n,n+1} + z_n^2 q_{n+1,n+1} = \frac{144n^{11}+3192n^{10}+31216n^9+177540n^8+651210n^7+1613062n^6}{(n+1)^3(n+3)(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2} + \frac{2743061n^5+3186210n^4+2460693n^3+1192988n^2+323673n+37086}{(n+1)^3(n+3)(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2}$$

and $s_n = q_{n+1,n} - z_n q_{n+1,n+1} = -\frac{(n+3)(2n+3)(2n^2+10n+11)(3n^2+7n+3)}{(n+1)(n+2)(n+4)(2n+5)(2n+7)(3n^2+13n+13)}$ when $0 \le n \le N-1$; and

$$d_N = \frac{60N^7+600N^6+2488N^5+5476N^4+6795N^3+4650N^2+1584N+207}{30(N+1)^3(N+2)^3(N+3)(2N+5)}.$$

Note that $det\ Y_N = det\ Q'_N = det\ Q_N$. Next we transform $Y_N$ into a triangular matrix with the same determinant, and we find that the new matrix has diagonal entries $\delta_n$ which are given by the recursion formula: $\delta_0 = d_0$, $\delta_n = d_n - s_{n-1}^2/\delta_{n-1}$ $(1 \le n \le N)$. An induction argument shows that

$$\delta_n \ge \frac{30(n+3)^6(2n+3)^2(2n^2+10n+11)^2(3n^2+7n+3)^2}{(n+1)^2(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2 g(n)} > 0, \text{ where}$$

$$g(n) = 60n^7 + 1020n^6 + 7348n^5 + 29016n^4 + 67679n^3 + 93031n^2 + 69633n + 21860$$

for $0 \le n \le N-1$; note that $g(n)$ is the numerator obtained in $d_N$ when $N$ is replaced by $n+1$. Since $d_N$ departs from the pattern set by the earlier $d_n$'s, $\delta_N$ must be handled separately:

$$\delta_N \ge \frac{60N^7+600N^6+2488N^5+5476N^4+6795N^3+4650N^2+1584N+207}{30(N+1)^3(N+2)^4(N+3)(2N+5)} > 0.$$

Therefore $\det Q_N = \prod_{j=0}^{N} \delta_j > 0$, and the proof is complete. □

For the induction step in the proof above, the initial estimate for $\delta_n$ came from computing $\frac{s_{N-1}^2}{d_N}$ and then replacing $N$ by $n+1$. From there, an adjustment was needed.

The verification of the induction step reduces to showing that a $19^{th}$ degree polynomial is positive for all $n \geq 1$. Below is the command that was given to SAGE to execute.

$n = \text{var}\ ('n')$

$\text{expand}((30*(n+2)\hat{}4*(144*n\hat{}11+3192*n\hat{}10+31216*n\hat{}9+177540*n\hat{}8+651210*n\hat{}7+1613062*n\hat{}6+2743061*n\hat{}5+3186210*n\hat{}4+2460693*n\hat{}3+1192988*n\hat{}2+323673*n+37086)-(n+1)*(n+4)*(2*n+5)*(2*n+7)*(3*n\hat{}2+13*n+13)\hat{}2*(60*(n-1)\hat{}7+1020*(n-1)\hat{}6+7348*(n-1)\hat{}5+29016*(n-1)\hat{}4+67679*(n-1)\hat{}3+93031*(n-1)\hat{}2+69633*(n-1)+21860))*(60*n\hat{}7+1020*n\hat{}6+7348*n\hat{}5+29016*n\hat{}4+67679*n\hat{}3+93031*n\hat{}2+69633*n+21860)-900*(n+1)*(n+2)\hat{}4*(n+3)\hat{}7*(2*n+3)\hat{}2*(2*n\hat{}2+10*n+11)\hat{}2*(3*n\hat{}2+7*n+3)\hat{}2)$

And this is the resulting SAGE worksheet output, which we denote by $f(n)$.

$f(n) = 220320*n\hat{}19+8325216*n\hat{}18+147344112*n\hat{}17+1621610588*n\hat{}16+12423804832*n\hat{}15+70274637076*n\hat{}14+303640886360*n\hat{}13+1022365685883*n\hat{}12+2710505167956*n\hat{}11+5672704072899*n\hat{}10+9319440019836*n\hat{}9+11820506702133*n\hat{}8+11159132582690*n\hat{}7+7175130478741*n\hat{}6+2225790478822*n\hat{}5-894429232807*n\hat{}4-1475079085458*n\hat{}3-812545969449*n\hat{}2-226952537400*n-26586925200$

$f(0) = -26586925200$

$f(1) = 48058098267150$

$f(2) = 29447930357308764$

$f(3) = 2740303120043884194$

$f(4) = 100611201083636165760$

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