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A modification of generalized Baskakov-Kantorovich operators

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Abstract. In this paper, we give some direct results and weighted approximation properties for a modification of generalized Baskakov-Kantorovich operators.

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1. Introduction

The Baskakov operators, defined by V.A. Baskakov [7], and their Kantorovich type modification ([11], p.115) are given by

$$B_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad x\geq 0, n\in\mathbb{N}$$

and

$$V_n(f;x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad x\geq 0, n\in\mathbb{N},$$

respectively. In the literature there are many studies which include Baskakov operators, Baskakov-Kantorovich operators and their generalizations. Some of them are [1], [3]- [8], [10]- [13] and [16]- [27]. We now deal only with the works which are necessary for this paper. In the identity

$$(1-t)^{-x}e^{at}=\sum_{k=0}^{\infty} P_k(x,a)\frac{t^k}{k!},$$

where $a\geq 0$ is any constant and

$$P_k(x,a)=\sum_{i=0}^{k}\binom{k}{i}(x)_i a^{k-i}$$

with $(x)_0 = 1$, $(x)_i = x(x+1)\cdots(x+i-1)$ for $i \geq 1$ by setting $x = n$ and $t = \frac{x}{1+x}$, Miheşan [18] constructed the generalized Baskakov operators

$$B_n^a(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}$$

for every $f \in C[0,\infty)$. He showed that these operators converge uniformly on $[0,b]$ for functions having exponential growth on positive x-axis and obtained the order of approximation with the help of the usual modulus of continuity. After that in [25], by proposing integral type modification of the operators B_n^a in the sense of Kantorovich as follows:

$$V_n^a(f;x) = ne^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N}$$

Wafi and Khatoon proved a Voronovskaya type theorem in polynomial weight spaces for these operators. Note that for $a = 0$ the operators B_n^a and V_n^a reduce to the operators B_n and V_n, respectively. In 2010, Erençin and Başcanbaz-Tunca [12] presented the following generalization of the operators $B_n^a(f;x)$

$$L_n(f;x) = e^{-\frac{a_n x}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n}\right) \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}, \tag{1.1}$$

where (a_n) are (b_n) are two sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0, \quad \lim_{n\to\infty} \frac{1}{b_n} = 0,$$

and investigated approximation properties of such operators by means of the weighted Korovkin type theorem given in [14, 15] and also introduced an application to functional differential equations which gives a recurrence relation for the monomials of that operators.

Very recently, Altomare, Montano and Leonessa [2] presented the modification of Szasz-Mirakyan-Kantorovich operators defined by

$$C_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left[\frac{n}{b_n - a_n} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)dt\right], \quad x \geq 0, n \in \mathbb{N},$$

where (a_n) and (b_n) are sequences of real numbers such that $0 \leq a_n < b_n \leq 1$. They introduced some approximation properties of these operators on continuous function spaces, weighted continuous function spaces and Lebesgue spaces and also obtained some estimates for the rate of convergence.

Inspired by that work, we consider the following Kantorovich type operators

$$K_n(f;x) = \sum_{k=0}^{\infty} S_{n,a_n}(k,x) \frac{b_n}{d_n - c_n} \int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N}, \tag{1.2}$$

where

$$S_{n,a_n}(k,x) = e^{-\frac{a_n x}{1+x}} \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}}$$

and (a_n), (b_n), (c_n) and (d_n) are sequences of real numbers having the properties:
(i) $a_n \geq 0, \quad b_n \geq 1, \quad 0 \leq c_n < d_n \leq 1$
(ii) $\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$
We remark that for $a_n = a$, $b_n = n$, $c_n = 0$ and $d_n = 1$ the operators $K_n(f;x)$ turn out to be the operators $V_n^a(f;x)$.

In the present paper, we first give some direct results. Next, we prove a weighted Korovkin type theorem and compute the order of approximation with the help of the weighted modulus of continuity for these operators.

2. Auxiliary results

By [12], we have

$$L_n(1;x) = 1 \tag{2.1}$$

$$L_n(t;x) = \frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} \tag{2.2}$$

$$L_n(t^2;x) = \frac{n(n+1)}{b_n^2}x^2 + \frac{2a_n n}{b_n^2}\frac{x^2}{1+x} + \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \frac{n}{b_n^2}x + \frac{a_n}{b_n^2}\frac{x}{1+x}, \tag{2.3}$$

where $L_n(f;x)$ is defined by (1.1).
In the sequel, we shall need to following lemmas.

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned} L_n(t^3;x) =& \frac{n(n+1)(n+2)}{b_n^3}x^3 + \frac{3a_n n(n+1)}{b_n^3}\frac{x^3}{1+x} + \frac{3a_n^2 n}{b_n^3}\frac{x^3}{(1+x)^2} \\ &+ \frac{a_n^3}{b_n^3}\frac{x^3}{(1+x)^3} + \frac{3n(n+1)}{b_n^3}x^2 + \frac{6a_n n}{b_n^3}\frac{x^2}{1+x} + \frac{3a_n^2}{b_n^3}\frac{x^2}{(1+x)^2} \\ &+ \frac{n}{b_n^3}x + \frac{a_n}{b_n^3}\frac{x}{1+x} \end{aligned}$$

and

$$\begin{aligned} L_n(t^4;x) =& \frac{n(n+1)(n+2)(n+3)}{b_n^4}x^4 + \frac{4a_n n(n+1)(n+2)}{b_n^4}\frac{x^4}{1+x} \\ &+ \frac{6a_n^2 n(n+1)}{b_n^4}\frac{x^4}{(1+x)^2} + \frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} \\ &+ \frac{6n(n+1)(n+2)}{b_n^4}x^3 + \frac{18a_n n(n+1)}{b_n^4}\frac{x^3}{1+x} + \frac{18a_n^2 n}{b_n^4}\frac{x^3}{(1+x)^2} \\ &+ \frac{6a_n^3}{b_n^4}\frac{x^3}{(1+x)^3} + \frac{7n(n+1)}{b_n^4}x^2 + \frac{14a_n n}{b_n^4}\frac{x^2}{1+x} + \frac{7a_n^2}{b_n^4}\frac{x^2}{(1+x)^2} \\ &+ \frac{n}{b_n^4}x + \frac{a_n}{b_n^4}\frac{x}{1+x}. \end{aligned}$$

It can be proved in a similar way that of the proof of Lemma 2.1 in [18] or by using the recurrence relation given in [12].

Lemma 2.2. *For the operators $K_n(f;x)$ defined by (1.2), we have*

$$\begin{aligned}
K_n(1;x) =& \ 1,\\
K_n(t;x) =& \ \frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\\
K_n(t^2;x) =& \ \frac{n(n+1)}{b_n^2}x^2 + \frac{2a_n n}{b_n^2}\frac{x^2}{1+x} + \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \frac{nm_1(n)}{b_n^2}x\\
&+ \frac{a_n m_1(n)}{b_n^2}\frac{x}{1+x} + \frac{m_2(n)}{3b_n^2},
\end{aligned}$$

$$\begin{aligned}
K_n(t^3;x) =& \frac{n(n+1)(n+2)}{b_n^3}x^3 + \frac{3a_n n(n+1)}{b_n^3}\frac{x^3}{1+x} + \frac{3a_n^2 n}{b_n^3}\frac{x^3}{(1+x)^2}\\
&+ \frac{a_n^3}{b_n^3}\frac{x^3}{(1+x)^3} + \frac{3n(n+1)m_3(n)}{2b_n^3}x^2 + \frac{3a_n n m_3(n)}{b_n^3}\frac{x^2}{1+x}\\
&+ \frac{3a_n^2 m_3(n)}{2b_n^3}\frac{x^2}{(1+x)^2} + \frac{nm_4(n)}{2b_n^3}x + \frac{a_n m_4(n)}{2b_n^3}\frac{x}{1+x} + \frac{m_5(n)}{4b_n^3},
\end{aligned}$$

and

$$\begin{aligned}
K_n(t^4;x) =& \frac{n(n+1)(n+2)(n+3)}{b_n^4}x^4 + \frac{4a_n n(n+1)(n+2)}{b_n^4}\frac{x^4}{1+x}\\
&+ \frac{6a_n^2 n(n+1)}{b_n^4}\frac{x^4}{(1+x)^2} + \frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4}\\
&+ \frac{2n(n+1)(n+2)m_6(n)}{b_n^4}x^3 + \frac{6a_n n(n+1)m_6(n)}{b_n^4}\frac{x^3}{1+x}\\
&+ \frac{6a_n^2 n m_6(n)}{b_n^4}\frac{x^3}{(1+x)^2} + \frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} + \frac{n(n+1)m_7(n)}{b_n^4}x^2\\
&+ \frac{2a_n n m_7(n)}{b_n^4}\frac{x^2}{1+x} + \frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} + \frac{nm_8(n)}{b_n^4}x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x}\\
&+ \frac{m_9(n)}{5b_n^4},
\end{aligned}$$

where
$m_0(n) = c_n + d_n$, $m_1(n) = c_n + d_n + 1$, $m_2(n) = c_n^2 + c_n d_n + d_n^2$,
$m_3(n) = c_n + d_n + 2$, $m_4(n) = 2(c_n^2 + c_n d_n + d_n^2) + 3(c_n + d_n) + 2$,
$m_5(n) = c_n^3 + c_n^2 d_n + c_n d_n^2 + d_n^3$, $m_6(n) = c_n + d_n + 3$,
$m_7(n) = 2(c_n^2 + c_n d_n + d_n^2) + 6(c_n + d_n) + 7$,
$m_8(n) = c_n^3 + c_n^2 d_n + c_n d_n^2 + d_n^3 + 2(c_n^2 + c_n d_n + d_n^2) + 2(c_n + d_n) + 1$ *and*
$m_9(n) = c_n^4 + c_n^3 d_n + c_n^2 d_n^2 + c_n d_n^3 + d_n^4$.

By using the definition of K_n, the equalities (2.1)- (2.3) and Lemma 2.1, it can be proved easily. So, we omit them.

Now in terms of the linearity of the operators K_n and Lemma 2.2 we can state the following lemma.

Lemma 2.3. *For the operators $K_n(f;x)$ defined by (1.2), we have*

$$\begin{aligned}K_n((t-x)^2;x) &= \left(\frac{n(n+1)}{b_n^2} - \frac{2n}{b_n} + 1\right)x^2 + \frac{2a_n}{b_n}\left(\frac{n}{b_n} - 1\right)\frac{x^2}{1+x} \\ &+ \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \left(\frac{nm_1(n)}{b_n^2} - \frac{m_0(n)}{b_n}\right)x \\ &+ \frac{a_n m_1(n)}{b_n^2}\frac{x}{1+x} + \frac{m_2(n)}{3b_n^2}\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&K_n((t-x)^4;x) \\ &= \left(\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right)x^4 \\ &+ \frac{4a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} - \frac{3n(n+1)}{b_n^2} + \frac{3n}{b_n} - 1\right)\frac{x^4}{1+x} \\ &+ \frac{6a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} - \frac{2n}{b_n} + 1\right)\frac{x^4}{(1+x)^2} + \frac{4a_n^3}{b_n^3}\left(\frac{n}{b_n} - 1\right)\frac{x^4}{(1+x)^3} \\ &+ \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} + 2\Bigg(\frac{n(n+1)(n+2)m_6(n)}{b_n^4} - \frac{3n(n+1)m_3(n)}{b_n^3} + \frac{3nm_1(n)}{b_n^2} \\ &- \frac{m_0(n)}{b_n}\Bigg)x^3 + \frac{6a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3} - \frac{2nm_3(n)}{b_n^2} + \frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x} \\ &+ \frac{6a_n^2}{b_n^2}\left(\frac{nm_6(n)}{b_n^2} - \frac{m_3(n)}{b_n}\right)\frac{x^3}{(1+x)^2} + \frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} \\ &+ \left(\frac{n(n+1)m_7(n)}{b_n^4} - \frac{2nm_4(n)}{b_n^3} + \frac{2m_2(n)}{b_n^2}\right)x^2 \\ &+ \frac{2a_n}{b_n}\left(\frac{nm_7(n)}{b_n^3} - \frac{m_4(n)}{b_n^2}\right)\frac{x^2}{1+x} + \frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} \\ &+ \left(\frac{nm_8(n)}{b_n^4} - \frac{m_5(n)}{b_n^3}\right)x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x} + \frac{m_9(n)}{5b_n^4},\end{aligned} \tag{2.5}$$

where $m_0(n)$,$m_1(n)$, $m_2(n)$, $m_3(n)$, $m_4(n)$,$m_5(n)$, $m_6(n)$, $m_7(n)$, $m_8(n)$ and $m_9(n)$ are given as in Lemma 2.2.

Lemma 2.4. *For the operators $K_n(f;x)$ defined by (1.2), we have*

$$K_n((t-x)^4;x) \le 12m_7(n)A(n)(x^4 + x^3 + x^2 + x + 1),$$

where $m_7(n)$ given as in Lemma 2.2 and $A(n) = \max\{A_1(n), A_2(n)\}$ with

$$\begin{aligned}A_1(n) = &\left|\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right| \\ &+ \frac{a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} + \frac{n}{b_n}\right) + \frac{a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} + 1\right) + \frac{a_n^3}{b_n^3}\left(\frac{n}{b_n} + \frac{a_n}{b_n}\right),\end{aligned}$$

$$A_2(n)=\frac{n(n+1)(n+2)}{b_n^4}+\frac{n}{b_n^2}+\frac{a_n}{b_n^2}\left(\frac{n(n+1)}{b_n^2}+1\right)+\frac{a_n^2}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right).$$

Proof. From (2.5) we may write

$$\begin{aligned}
&K_n((t-x)^4;x)\\
\le&\left(\frac{n(n+1)(n+2)(n+3)}{b_n^4}-\frac{4n(n+1)(n+2)}{b_n^3}+\frac{6n(n+1)}{b_n^2}-\frac{4n}{b_n}+1\right)x^4\\
&+\frac{4a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3}+\frac{3n}{b_n}\right)\frac{x^4}{1+x}+\frac{6a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2}+1\right)\frac{x^4}{(1+x)^2}\\
&+\frac{4a_n^3n}{b_n^4}\frac{x^4}{(1+x)^3}+\frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4}+2\left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4}+\frac{3nm_1(n)}{b_n^2}\right)x^3\\
&+\frac{6a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3}+\frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x}+\frac{a_n^2nm_6(n)}{b_n^4}\frac{x^3}{(1+x)^2}\\
&+\frac{2a_n^3m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3}+\left(\frac{n(n+1)m_7(n)}{b_n^4}+\frac{2m_2(n)}{b_n^2}\right)x^2+\frac{2a_nnm_7(n)}{b_n^4}\frac{x^2}{1+x}\\
&+\frac{a_n^2m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2}+\frac{nm_8(n)}{b_n^4}x+\frac{a_nm_8(n)}{b_n^4}\frac{x}{1+x}+\frac{m_9(n)}{5b_n^4}\\
\le&12\Bigg\{\left|\frac{n(n+1)(n+2)(n+3)}{b_n^4}-\frac{4n(n+1)(n+2)}{b_n^3}+\frac{6n(n+1)}{b_n^2}-\frac{4n}{b_n}+1\right|x^4\\
&+\frac{a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3}+\frac{n}{b_n}\right)\frac{x^4}{1+x}+\frac{a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2}+1\right)\frac{x^4}{(1+x)^2}\\
&+\frac{a_n^3n}{b_n^4}\frac{x^4}{(1+x)^3}+\frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4}+\left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4}+\frac{nm_1(n)}{b_n^2}\right)x^3\\
&+\frac{a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3}+\frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x}+\frac{a_n^2nm_6(n)}{b_n^4}\frac{x^3}{(1+x)^2}\\
&+\frac{a_n^3m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3}+\left(\frac{n(n+1)m_7(n)}{b_n^4}+\frac{m_2(n)}{b_n^2}\right)x^2+\frac{a_nnm_7(n)}{b_n^4}\frac{x^2}{1+x}\\
&+\frac{a_n^2m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2}+\frac{nm_8(n)}{b_n^4}x+\frac{a_nm_8(n)}{b_n^4}\frac{x}{1+x}+\frac{m_9(n)}{b_n^4}\Bigg\}.
\end{aligned}$$

Since $\frac{x^s}{(1+x)^l}\le x^s$ for all $x\ge 0$, $l\le s$ $(l,s=1,2,3,4)$ and $m_1(n)$, $m_2(n)$, $m_6(n)$, $m_8(n)$, $m_9(n)<m_7(n)$ for all $n\in\mathbb{N}$ one gets

$$K_n((t-x)^4;x)\le 12\Big\{A_1(n)x^4+m_7(n)\left[A_2(n)x^3+A_3(n)x^2+A_4(n)x+A_5(n)\right]\Big\},$$

where

$$A_3(n)=\frac{n(n+1)}{b_n^4}+\frac{1}{b_n^2}+\frac{a_n}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right),A_4(n)=\frac{1}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right),A_5(n)=\frac{1}{b_n^4}.$$

Finally, since $A_5(n) \leq A_4(n) < A_3(n) < A_2(n)$ for all $n \in \mathbb{N}$ we can write

$$K_n((t-x)^4; x) \leq 12\Big\{ A_1(n)x^4 + m_7(n)A_2(n)\left(x^3+x^2+x+1\right)\Big\}$$
$$\leq 12m_7(n)\Big\{ A_1(n)x^4 + A_2(n)\left(x^3+x^2+x+1\right)\Big\}$$

which gives the desired result. □

3. Direct results

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on the interval $[0,\infty)$, endowed with the norm

$$\|f\| = \sup_{0\leq x<\infty} |f(x)|.$$

For any $\delta > 0$, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\| + \delta\,\|g''\|\right\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz ([9], p.177, Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{3.1}$$

where the second order modulus of smoothness of $f \in C_B[0,\infty)$ is defined as

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\ \sup_{0\leq x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f;\delta) = \sup_{0<h\leq\delta}\ \sup_{0\leq x<\infty} |f(x+h) - f(x)|.$$

Now consider the following operator

$$\widehat{K}_n(f;x) = K_n(f;x) - f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) + f(x),$$

where $m_0(n)$ given as in Lemma 2.2.

Lemma 3.1. *Let $g \in C_B^2[0,\infty)$. Then we have*

$$\left|\widehat{K}_n(g;x) - g(x)\right| \leq \delta_n(x)\|g''\|,$$

where

$$\delta_n(x) = K_n((t-x)^2;x) + \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2.$$

Proof. By the definition of the operators $\widehat{K}_n$ and Lemma 2.2 we get

$$\begin{aligned}\widehat{K}_n(t-x;x) &= K_n(t-x,x) - \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - x\right) \\ &= K_n(t,x) - xK_n(1,x) - \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - x\right) \\ &= 0.\end{aligned}$$

Let $g \in C_B^2[0,\infty)$ and $x \in [0,\infty)$. By Taylor's formula of g

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty)$$

one may write

$$\begin{aligned}&\widehat{K}_n(g;x) - g(x) \\ =&g'(x)\widehat{K}_n(t-x,x) + \widehat{K}_n\left(\int_x^t (t-u)g''(u)du;x\right) \\ =&\widehat{K}_n\left(\int_x^t (t-u)g''(u)du;x\right) \\ =&K_n\left(\int_x^t (t-u)g''(u)du;x\right) \\ &- \int_x^{\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}} \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - u\right) g''(u)du.\end{aligned}$$

Now using the following inequalities

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \left\|g''\right\|$$

and

$$\begin{aligned}&\left|\int_x^{\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}} \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - u\right) g''(u)du\right| \\ &\leq \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2 \left\|g''\right\|\end{aligned}$$

we reach to

$$\begin{aligned}&\left|\widehat{K}_n(g;x) - g(x)\right| \\ &\leq \left\{K_n((t-x)^2;x) + \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2\right\} \left\|g''\right\| \\ &= \delta_n(x)\|g''\|.\end{aligned}$$

□

Theorem 3.2. *Let $f \in C_B[0,\infty)$. Then for all $x \in [0,\infty)$ there exists a constant $A > 0$ such that*

$$|K_n(f;x) - f(x)| \leq A\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right),$$

where $\delta_n(x)$ defined as in Lemma 3.1.

Proof. By means of the definitions of the operators $\widehat{K}_n$ and K_n we have

$$\begin{aligned}|K_n(f;x) - f(x)| \leq &\left|\widehat{K}_n(f-g;x)\right| + |(f-g)(x)| + \left|\widehat{K}_n(g;x) - g(x)\right| \\ &+ \left|f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) - f(x)\right|\end{aligned}$$

and

$$\left|\widehat{K}_n(f;x)\right| \leq |K_n(f;x)| + 2\|f\| \leq \|f\|K_n(1;x) + 2\|f\| = 3\|f\|.$$

Thus we may conclude that

$$\begin{aligned}|K_n(f;x) - f(x)| \leq &4\|f-g\| + \left|\widehat{K}_n(g;x) - g(x)\right| \\ &+ \left|f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) - f(x)\right|.\end{aligned}$$

In the light of Lemma 3.1 one gets

$$\begin{aligned}|K_n(f;x) - f(x)| \leq &4\|f-g\| + \delta_n(x)\|g''\| \\ &+ \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right).\end{aligned}$$

Therefore taking the infimum over all $g \in C_B^2[0,\infty)$ on the right-hand side of the last inequality and considering (3.1), we find that

$$\begin{aligned}|K_n(f;x) - f(x)| &\leq 4K_2(f;\delta_n(x)) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) \\ &\leq 4C\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) \\ &= A\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right)\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let $0 < \gamma \leq 1$ and $f \in C_B[0,\infty)$. Then if $f \in Lip_M(\gamma)$, that is, the inequality*

$$|f(t) - f(x)| \leq M\,|t-x|^\gamma, \quad x,t \in [0,\infty)$$

holds, then for each $x \in [0,\infty)$ we have

$$|K_n(f;x) - f(x)| \leq \delta_n^{\frac{\gamma}{2}}(x),$$

where $\delta_n(x) = K_n((t-x)^2;x)$ and $M > 0$ is a constant.

Proof. Let $f \in C_B[0,\infty) \cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators K_n we get

$$\begin{aligned} |K_n(f;x) - f(x)| &\leq K_n\left(|f(t) - f(x)|\,; x\right) \\ &\leq MK_n\left(|t-x|^{\gamma}\,; x\right) \\ &= M\sum_{k=0}^{\infty} S_{n,a_n}(k,x)\frac{b_n}{d_n - c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} |t-x|^{\gamma}\,dt. \end{aligned}$$

Now applying the Hölder inequality two times successively with $p = \frac{2}{\gamma}$, $q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned} |K_n(f;x) - f(x)| &\leq M\sum_{k=0}^{\infty} S_{n,a_n}(k,x)\left\{\frac{b_n}{d_n - c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} (t-x)^2 dt\right\}^{\frac{\gamma}{2}} \\ &\leq MK_n\left((t-x)^2; x\right)^{\frac{\gamma}{2}} \\ &= M\delta_n^{\frac{\gamma}{2}}(x). \end{aligned}$$

This completes the proof. □

4. Weighted approximation properties

Now we introduce convergence properties of the operators K_n via the weighted Korovkin type theorem given by Gadjiev in [14, 15]. For this purpose we recall some definitions and notations.

Let $\rho(x) = 1 + x^2$ and $B_\rho[0,\infty)$ be the space of all functions having the property

$$|f(x)| \leq M_f\rho(x),$$

where $x \in [0,\infty)$ and M_f is a positive constant depending only on f. $B_\rho[0,\infty)$ is equipped with the norm

$$\|f\|_\rho = \sup_{0 \leq x < \infty} \frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$. By $C_\rho^0[0,\infty)$ we denote the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

Theorem A [14, 15]**:** *Let $\{A_n\}$ be a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ and satisfying the conditions*

$$\lim_{n\to\infty} ||A_n(t^\nu; x) - x^\nu||_\rho = 0, \quad \nu = 0,1,2.$$

Then for any function $f \in C_\rho^0[0,\infty)$,

$$\lim_{n\to\infty} ||A_n(f;x) - f(x)||_\rho = 0.$$

Note that a sequence of linear positive operators A_n acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$||A_n(\rho;x)||_\rho \le M\rho,$$

where $M\rho$ is positive constant. This fact is a simple result of the necessary and sufficient condition that

$$A_n(\rho;x) \le M\rho(x)$$

given in [14, 15].

Theorem 4.1. *Let $\{K_n\}$ be the sequence of linear positive operators defined by (1.2). Then for each $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty} ||K_n(f;x) - f(x)||_\rho = 0.$$

Proof. Using Lemma 2.2, we may write

$$\sup_{0\le x<\infty} \frac{|K_n(\rho;x)|}{1+x^2} \le 1 + \frac{n(n+1)}{b_n^2} + \frac{2a_n n}{b_n^2} + \frac{a_n^2}{b_n^2} + \frac{nm_1(n)}{b_n^2} + \frac{a_n m_1(n)}{b_n^2} + \frac{m_2(n)}{3b_n^2}.$$

Since $\lim_{n\to\infty} \frac{n}{b_n} = 1$ we have $\lim_{n\to\infty} \frac{1}{b_n} = 0$. Thus under the conditions (i) and (ii), there exists a positive constant M^* such that

$$\frac{n(n+1)}{b_n^2} + \frac{2a_n n}{b_n^2} + \frac{a_n^2}{b_n^2} + \frac{nm_1(n)}{b_n^2} + \frac{a_n m_1(n)}{b_n^2} + \frac{m_2(n)}{3b_n^2} < M^*$$

for each n. Hence we get

$$||K_n(\rho;x)||_\rho \le 1 + M^*$$

which shows that $\{K_n\}$ is a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

In order to complete the proof, it is enough to prove that the conditions of Theorem A

$$\lim_{n\to\infty} ||K_n(t^\nu;x) - x^\nu||_\rho = 0, \quad \nu = 0,1,2$$

are satisfied. It is clear that

$$\lim_{n\to\infty} ||K_n(1;x) - 1||_\rho = 0.$$

By Lemma 2.2, we have

$$\begin{aligned} ||K_n(t;x) - x||_\rho &= \sup_{0\le x<\infty} \left| \left(\frac{n}{b_n} - 1\right) \frac{x}{1+x^2} + \frac{a_n}{b_n} \frac{x}{(1+x)(1+x^2)} + \frac{m_0(n)}{2b_n} \frac{1}{1+x^2} \right| \\ &\le \left| \frac{n}{b_n} - 1 \right| + \frac{a_n}{b_n} + \frac{m_0(n)}{b_n}. \end{aligned}$$

Thus taking into consideration the conditions (i) and (ii) we can conclude that

$$\lim_{n\to\infty} ||K_n(t;x) - x||_\rho = 0.$$

Similarly, one gets

$$\begin{aligned}
&||K_n(t^2;x)-x^2||_\rho \\
&= \sup_{0\le x<\infty}\left|\left(\frac{n(n+1)}{b_n^2}-1\right)\frac{x^2}{1+x^2}+\frac{2a_n n}{b_n^2}\frac{x^2}{(1+x)(1+x^2)}+\frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2(1+x^2)}\right. \\
&\quad \left.+\frac{nm_1(n)}{b_n^2}\frac{x}{1+x^2}+\frac{a_n m_1(n)}{b_n^2}\frac{x}{(1+x)(1+x^2)}+\frac{m_2(n)}{3b_n^2}\frac{1}{(1+x^2)}\right| \\
&\le \left|\frac{n(n+1)}{b_n^2}-1\right|+\frac{2a_n n}{b_n^2}+\frac{a_n^2}{b_n^2}+\frac{nm_1(n)}{b_n^2}+\frac{a_n m_1(n)}{b_n^2}+\frac{m_2(n)}{b_n^2}
\end{aligned}$$

which leads to

$$\lim_{n\to\infty}||K_n(t^2;x)-x^2||_\rho=0.$$

Thus the proof is completed. □

Now we compute the order of approximation of the operators K_n in terms of the weighted modulus of continuity $\Omega_2(f,\delta)$ (see[17]) defined by

$$\Omega_2(f,\delta)=\sup_{x\ge 0,0<h\le\delta}=\frac{|f(x+h)-f(x)|}{1+(x+h)^2},\quad f\in C_\rho^0[0,\infty)$$

and has the following properties:
(a) $\Omega_2(f,\delta)$ is monotone increasing function of δ,
(b) $\lim_{\delta\to 0^+}\Omega_2(f,\delta)=0$,
(c) for each $\lambda\in\mathbb{R}^+,\Omega_2(f,\lambda\delta)\le(\lambda+1)\Omega_2(f,\delta)$.

Theorem 4.2. *Let $\{K_n\}$ be the sequence of linear positive operators defined by (1.2). Then for each $f\in C_\rho^0[0,\infty)$, we have*

$$\sup_{0\le x<\infty}\frac{|K_n(f;x)-f(x)|}{(1+x^2)^3}\le C\Omega_2\left(f,[m_7(n)A(n)]^{\frac{1}{4}}\right),$$

where C is positive constant and $m_7(n)$ and $A(n)$ defined as in Lemma 2.2 and Lemma 2.4, respectively.

Proof. For $x\ge 0$ and $t\ge 0$, by the definition of $\Omega_2(f,\delta)$ and the property (c), we may write

$$\begin{aligned}
|f(t)-f(x)| &\le \left(1+(x+|t-x|)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n) \\
&\le 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n).
\end{aligned}$$

By using the monotonicity of K_n and the following inequality (see [16])

$$\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\le 2(1+\delta_n^2)\left(1+\frac{(t-x)^4}{\delta_n^4}\right)$$

one gets

$$
\begin{aligned}
|K_n(f;x)-f(x)| &\leq 2(1+x^2)K_n\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right);x\right)\Omega_2(f,\delta_n)\\
&\leq 4(1+\delta_n^2)(1+x^2)K_n\left(1+\frac{(t-x)^4}{\delta_n^4};x\right)\Omega_2(f,\delta_n)\\
&= 4(1+\delta_n^2)(1+x^2)\left[1+\frac{1}{\delta_n^4}K_n((t-x)^4;x)\right]\Omega_2(f,\delta_n)\\
&\leq C_1(1+x^2)\left[1+\frac{1}{\delta_n^4}K_n((t-x)^4;x)\right]\Omega_2(f,\delta_n).
\end{aligned}
$$

With the help of the Lemma 2.4 this inequality leads to

$$
|K_n(f;x)-f(x)| \leq 12C_1(1+x^2)\left[1+\frac{m_7(n)A(n)}{\delta_n^4}(x^4+x^3+x^2+x+1)\right]\Omega_2(f,\delta_n)
$$

which gives the required result. □

We observe that in Theorem 4.1 we have showed that K_n converges to f in the weighted space $C\rho[0,\infty)$. But in Theorem 4.2 we have computed the rate of convergence for these operators in the weighted space $C\rho^3[0,\infty)$.

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