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Restricting the Clifford extensions of a pointed group

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Abstract. In this note we give a restriction of the isomorphism constructed in the main result of [1], which states that the Clifford extensions of two Brauer correspondent points are isomorphic, by using a defect pointed group instead of an ordinary defect group and by replacing the Brauer quotient with the multiplicity algebra of the mentioned pointed group.

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1. Introduction

Let p be a prime and $\mathcal{O}$ a discrete valuation ring such that k is the residue field of $\mathcal{O}$. We also consider a finite group G and a normal subgroup N of G.

An N-interior G-algebra is an $\mathcal{O}$-algebra A endowed with two group homomorphisms

$$N \to A^* \text{ and } G \to \mathrm{Aut}_{\mathcal{O}}(A).$$

We denote by A^* the group of invertible elements of A. Then

$$N \ni y \mapsto y \cdot 1 = 1 \cdot y \in A^*$$

and

$$G \ni x \mapsto \varphi(x) \in \mathrm{Aut}_{\mathcal{O}}(A);$$

i.e. any x determines an $\mathcal{O}$-algebra automorphism of A. We use standard notation impling conjugation on the right:

$$\varphi(x)(a) =: a^x \text{ and } a^y := y^{-1} \cdot a \cdot y,$$

for any $x \in G$, $y \in N$ and $a \in A$.

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Let H by any subgroup of G. A point α of H on A, denoted H_α, is a $(A^H)^*$-conjugacy class of a primitive idempotent $i \in A^H$. Throughout we use notations as: $H_{\{\alpha\}}$-denoting the stabilizer of α, provided that H acts on the subalgebra of A that contains the point α; H_i-denoting the stabilizer of some idempotent i and also $N_H(K_\alpha)$-the subgroup of H that normalizes the group K and stabilizes the point α.

We use [5, Theorem 8.20] to characterize fusions on an N-interior G-algebra A. Let α be a point of A^H and let $j \in \alpha$. Any element $x \in N_G(H)$ acts by conjugation on H and determines an automorphism. If this automorphism satisfies $y^{-1}y^{x^{-1}} \in N$ for any $y \in H$, then it determines an A-fusion from the pointed group H_α to itself if and only if there exists $a \in A^*$ such that for any $y \in H$ we have

$$aj = ja, \qquad a^y \cdot y^{-1}y^{x^{-1}} = a.$$

We have already introduced in [1] the so-called *Clifford extensions* of points. Constructing such an extension implies working with an N-interior G-algebra A, a point β of N on A, and with P, a defect group of β that is contained in N. Let $\bar{\beta}$ denote the correspondent point of β determined by the Brauer morphism. The main result of [1] states that the extension corresponding to β is isomorphic to the extension corresponding to $\bar{\beta}$. This result generalizes the main result of [2, Section 12].

In this paper we construct an analogous isomorphism of extensions by using a defect pointed group P_γ of β and by replacing the Brauer quotient of A^P with the multiplicity algebra of P_γ. The first replacement forces a new computation of the groups that appear in original construction of the Clifford extension of β. The second replacement, that is the replacement made with regard to the Brauer quotient, generates a slightly more complicated situation with respect to the gradings. It seems that the grading of the new quotient, that contains the multiplicity algebra as the identity component, depends on the units of a source algebra of the pointed group N_β.

We use standard notations and we refer the reader to [4] and [6] for details regarding the theory of G-algebras and pointed groups.

2. Existing constructions and results

2.1. Let us recall the notations and quote the existing results. For more details regarding the proofs of the following statements we refer to [1]. Let A be an N-interior G-algebra, P be a p-subgroup of N, and let $\beta \in \mathcal{P}(A^N|P_\gamma)$, that is a point of N on A with defect pointed group P_γ. We denote by $\bar{G}$ the quotient group G/N. Following [1, Proposition 3.5] we see that the action of G on A^N gives rise to the normalizer $N_G(N_\beta)$ which satisfies

$$N_G(N_\beta) = N_G(P)_{\{\beta\}}N.$$

2.2. Consider the algebra

$$\hat{A} = A \otimes_N G = \bigoplus_x A \otimes x,$$

where x runs through a set of representatives of the classes of $\bar{G}$. The product in $\hat{A}$ is given by

$$(a \otimes x)(b \otimes y) = ab^{x^{-1}} \otimes xy.$$

Then $\hat{A}$ is clearly a G-interior algebra via the following morphism of groups

$$G \to A^*, \text{ where } g \mapsto 1 \otimes g.$$

2.3. If $j \in \beta$, due to the action of $N_G(N_\beta)$ on A^N, for any $x \in N_G(N_\beta)$ we have $j^x = aja^{-1}$, for some $a \in (A^N)^*$. This allows the construction of the following two groups (since the group $A^* \otimes G < \hat{A}^*$ acts by conjugation on $\hat{A}$):

$$\widehat{N} := (A^N)^*_j \otimes N$$

and

$$\widehat{N_G(N_\beta)} := ((A^N)^* \otimes N_G(N_\beta))_j.$$

The exact sequence

$$1 \to \widehat{N} \to \widehat{N_G(N_\beta)} \to \bar{N}_G(N_\beta) \to 1 \tag{i}$$

induces the isomorphism

$$\widehat{N_G(N_\beta)}/\widehat{N} \simeq \bar{N}_G(N_\beta) = N_G(N_\beta)/N.$$

Now $A_\beta := jAj$ is an $\widehat{N}$-interior $\widehat{N_G(N_\beta)}$-algebra, it is actually N-interior. Next we see that

$$\hat{A}_\beta := A_\beta \otimes_{\widehat{N}} \widehat{N_G(N_\beta)}$$

is an N-interior algebra, thus we consider the set

$$\bar{G}[\beta] = \{\bar{x} \in \bar{N}_G(N_\beta) \mid (A_\beta \otimes \hat{x})^N \cdot (A_\beta \otimes \hat{x}^{-1})^N = (A_\beta)^N\},$$

where $\hat{x}$ is a lifting of $\bar{x}$ via (i). This set turns out to be a normal subgroup of $\bar{N}_G(N_\beta)$, see [1, Proposition 2.7].

2.4. Let $N_G^A(N_\beta)$ denote the group consisting of elements $x \in N_G(N_\beta)$ such that the conjugation action of x on N induces an A-fusion from N_β to itself. We have

$$\bar{G}[\beta] \simeq \bar{N}_G^A(N_\beta).$$

Set $\widehat{N_G^A(N_\beta)}$, the inverse image of $\bar{G}[\beta]$ in $\widehat{N_G(N_\beta)}$, and set

$$\hat{A}[\beta] := A_\beta \otimes_{\widehat{N}} \widehat{N_G^A(N_\beta)}.$$

The algebra

$$\hat{A}[\beta]^N := \bigoplus_x (A_\beta \otimes \hat{x})^N,$$

where x runs through a set of representatives for the classes in $\bar{G}[\beta]$, is a crossed product and it is also an $\widehat{N_G(N_\beta)}$-algebra, since N and $\widehat{N_G^A(N_\beta)}$ are two normal subgroups of $\widehat{N_G(N_\beta)}$. The quotient

$$\hat{\bar{A}}[\beta]^N := \hat{A}[\beta]^N / J_{gr}(\hat{A}[\beta]^N)$$

is the twisted group algebra of $\hat{k} := A_\beta(N_\beta) = A_\beta^N/J(A_\beta^N)$ with $\bar{G}[\beta]$, actually a crossed product of $\hat{k}$ with $\bar{G}[\beta]$, and it corresponds uniquely to the Clifford extension of β

$$1 \to \hat{k}^* \to \widehat{\bar{N}_G^A(N_\beta)} \to \bar{G}[\beta] \to 1. \tag{1}$$

We denoted by J_{gr} the graded Jacobson radical. The qoutient $\hat{k}$ is a skew field whose center is an extension of k and $\widehat{\bar{N}_G^A(N_\beta)}$ denotes the group of homogeneous units of $\hat{\bar{A}}[\beta]^N$.

3. The Clifford extension of the multiplicity algebra

3.1. Consider $\beta \subseteq A^N$ as above, having defect pointed group P_γ. Since γ is a point of A^P, it corresponds uniquely to the $N_G(P_\gamma)$-invariant maximal ideal m_γ of A^P such that $\gamma \nsubseteq m_\gamma$. Consider the multiplicity algebra of P_γ :

$$A(P_\gamma) := A^P/m_\gamma.$$

The map

$$s_\gamma : A^P \to A(P_\gamma)$$

is an epimorphism of $N_G(P_\gamma)$-algebras. The restriction

$$s_\gamma : A_P^N \to A(P_\gamma)_P^{N_N(P_\gamma)}$$

is still an epimorphism of $N_G(P_\gamma)$-algebras and $\bar{\beta} := s_\gamma(\beta) \subseteq A(P_\gamma)^{N_N(P_\gamma)}$ is the unique correspondent point of β having P as a defect group, see [5, Theorem 6.14]. Set $\bar{j} := s_\gamma(j)$.

3.2. Denote by $N_G^A(P_\gamma)$ the subgroup of $N_G(P_\gamma)$ consisting of elements that determine A-fusions from P_γ to itself. Then

$$\bar{G}[\beta]_{\{\gamma\}} := \bar{G}[\beta] \cap \bar{N}_G^A(P_\gamma)$$

is the subgroup of $\bar{N}_G(N_\beta)$ such that any representative of any class determines A-fusions for both N_β and P_γ.

Denote by $\widehat{N_G^A(N_\beta)}_{\{\gamma\}}$ the inverse image of $\bar{G}[\beta]_{\{\gamma\}}$ via (i). In this case the algebra $\hat{A}_\beta^\gamma := A_\beta \otimes_{\hat{N}} \widehat{N_G^A(N_\beta)}_{\{\gamma\}}$ is N-invariant, hence

$$(\hat{A}_\beta^\gamma)^P := (A_\beta \otimes_{\hat{N}} \widehat{N_G^A(N_\beta)}_{\{\gamma\}})^P = \bigoplus_{\hat{x}} (A_\beta \otimes \hat{x})^P,$$

where $\hat{x}$ lifts a set of representatives of the classes in $\bar{G}[\beta]_{\{\gamma\}}$, is strongly $\bar{G}[\beta]_{\{\gamma\}}$-graded, it is actually a crossed-product.

Proposition 3.3. *With the above notations*

$$\widehat{m}_\gamma := m_\gamma \cdot (\hat{A}_\beta^\gamma)^P = (\hat{A}_\beta^\gamma)^P \cdot m_\gamma$$

is a two-sided ideal of $(\hat{A}_\beta^\gamma)^P$.

Proof. It suffices to prove the equality

$$m_\gamma \cdot (A_\beta \otimes \hat{x})^P = (A_\beta \otimes \hat{x})^P \cdot m_\gamma, \tag{*}$$

for any lifting $\hat{x}$.

The inclusion $P_\gamma \leq N_\beta$ provides an idempotent $i \in \gamma$ such that $ji = i = ji$. Thus the inclusion $A_\gamma \subseteq A_\beta$ gives the following homomorphism of groups

$$\phi : A_\gamma^* \to A_\beta^*, \text{ where } A_\gamma \ni a \mapsto a' := a + j - i \in A_\beta.$$

Then, if $x \in N_G(N_\beta)_{\{\gamma\}}$ determines an A-fusion from P_γ to itself we get

$$y^{x^{-1}} = y^a,$$

for some $a \in A_\gamma^*$ and for any $y \in P$. Clearly $a'a^{-1} = i \in A^P$, then we get

$$a'a^{-1} \cdot y = y \cdot a'a^{-1},$$

or equivalently $y^a = y^{a'}$, for any $y \in P$. We obtain $(a')^y \cdot y^{-1}xyx^{-1} = a'$ and for a lifting $\hat{x}$ of $\bar{x}$ this is equivalent to

$$(a' \otimes \hat{x})^y = a' \otimes \hat{x},$$

for any $y \in P$, as a homogeneous unit of $(\hat{A}_\beta^\gamma)^P$. Using all the above we can view

$$(\hat{A}_\beta^\gamma)^P = \bigoplus_{\hat{x}} (a' \otimes \hat{x}) \cdot A_\beta^P,$$

where $\hat{x}$ lifts via extension (i) a set of representatives of the classes of $\bar{G}[\beta]_{\{\gamma\}}$. All homogeneous units $a' \otimes \hat{x}$ of $(\hat{A}_\beta^\gamma)^P$ satisfy $a' \in N_{A^*}(P)$ and $a'i = ia'$. Hence for proving $(*)$ it suffices to prove

$$(a' \otimes \hat{x})^{-1} \cdot m_\gamma \cdot (a' \otimes \hat{x}) = m_\gamma.$$

Since $\hat{x}$ lifts an element of $\bar{N}_G(P_\gamma)$ the last equality is equivalent to

$$(a')^{-1} \cdot m_\gamma \cdot a' = m_\gamma.$$

The maximal ideal $(a')^{-1} \cdot m_\gamma \cdot a'$ of A^P can not contain γ, because otherwise

$$\gamma^{a'} = \gamma \subseteq m_\gamma,$$

which is a contradiction. □

The next proposition follows from the proof of the above proposition.

Proposition 3.4. *The group $\bar{G}[\beta]_{\{\gamma\}}$ is isomorphic to the subgroup of $\bar{N}_G^A(N_\beta)$ that consists of elements $\bar{x}$ such that for any lifting $\hat{x}$ the module $(A_\beta \otimes \hat{x})^P$ contains a homogeneous unit $a' \otimes \hat{x}$ satisfying $a' \in \phi(A_\gamma^*)$.*

Proof. Indeed, any element $x \in N_G(P_\gamma)_{\{\beta\}}$ that determines A-fusions for both P_γ and N_β lifts to $\hat{x}$ which gives, for any $y \in P$,

$$y^{\hat{x}^{-1}} = y^a = y^{a'},$$

for some $a \in A_\gamma^*$ and $a' = a + j - i$. □

3.5. In what follows it is more convenient to replace $\hat{N}$ with $(A^N)^*_j \otimes N_N(P_\gamma)$ and $\widehat{N_G(N_\beta)}$ with $((A^N)^* \otimes N_G(P_\gamma)_{\{\beta\}})_j$, since the two pairs give isomorphic quotients. We further denote

$$\widehat{N_G(N_\beta)}_{\{\gamma\}} := ((A^N)^* \otimes N_G(P_\gamma)_{\{\beta\}})_{j,i},$$

the subgroup of $((A^N)^* \otimes N_G(P_\gamma)_{\{\beta\}})_j$ whose elements also fix i. With this setting we have:

Lemma 3.6. *The group $\widehat{N_G^A(N_\beta)}_{\{\gamma\}}$ is $\widehat{N_G(N_\beta)}_{\{\gamma\}}$-invariant.*

Proof. Let $\hat{x} \in \widehat{N_G^A(N_\beta)}_{\{\gamma\}}$, $\hat{z} \in \widehat{N_G(N_\beta)}_{\{\gamma\}}$ and $a' \in \phi(A_\gamma^*)$ such that

$$(a' \otimes \hat{x})^y = a' \otimes \hat{x},$$

for any $y \in P$. Then $(a' \otimes \hat{x})^{\hat{z}} = (a')^{\hat{z}} \otimes \hat{x}^{\hat{z}}$ is also a homogeneous unit of $(A_\beta \otimes \hat{x}^{\hat{z}})^P$ verifying $\hat{x}^{\hat{z}} \in \widehat{N_G^A(N_\beta)}$ and $(a')^{\hat{z}} \in \phi(A_\gamma^*)$. ☐

3.7. Using the N-invariance of $\hat{A}_\beta^\gamma$ we obtain

$$\begin{aligned}(\bar{\hat{A}}_\beta^\gamma)^N &:= (\hat{A}_\beta^\gamma)^N / J_{gr}((\hat{A}_\beta^\gamma)^N) \\ &= \bigoplus_{\hat{x}} \left((A_\beta \otimes \hat{x})^N / J(A_\beta^N)(A_\beta \otimes \hat{x})^N\right),\end{aligned}$$

the crossed product of $\hat{k} = A_\beta(N_\beta)$ with $\bar{G}[\beta]_{\{\gamma\}}$, and simultaneously the twisted group algebra of $\hat{k}$ with $\bar{G}[\beta]_{\{\gamma\}}$ corresponding to the Clifford extension

$$1 \to \hat{k}^* \to \widehat{\bar{N}_G^A(N_\beta)}_{\{\gamma\}} \to \bar{G}[\beta]_{\{\gamma\}} \to 1. \tag{1'}$$

Clearly (1') is a subextension of (1).

3.8. Constructions similar to that of 2.3, making use of the action of $N_G(P_\gamma)$ on $A(P_\gamma)^{N_N(P_\gamma)}$, determine the exact sequence

$$\begin{aligned}1 &\to (A(P_\gamma)^{N_N(P_\gamma)})^*_{\bar{j}} \otimes N_N(P_\gamma) \to \\ &\to ((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar{\beta}\}})_{\bar{j}} \to \bar{N}_G(P_\gamma)_{\{\bar{\beta}\}} \to 1.\end{aligned} \tag{ii}$$

We set

$$\widehat{N_N(P_\gamma)} := ((A(P_\gamma)^{N_N(P_\gamma)})^*_{\bar{j}} \otimes N_N(P_\gamma))_{\bar{i}} \text{ and}$$

$$\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}} := ((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar{\beta}\}})_{\bar{j},\bar{i}},$$

where $\bar{i} = s_\gamma(i)$. One easily checks that $\widehat{N_N(P_\gamma)}$ is a normal subgroup in $\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$.

Lemma 3.9. *The following statements hold.*

a) *The groups $N_G(P_\gamma)_{\{\beta\}}$ and $N_G(P_\gamma)_{\{\bar{\beta}\}}$ coincide.*

b) *The groups $N_G(P_\gamma)_i$ and $N_G(P_\gamma)_{\bar{i}}$ coincide, hence $N_N(P_\gamma)_i = N_N(P_\gamma)_{\bar{i}}$ is a normal subgroup of $\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$ and of $\widehat{N_G(N_\beta)}_{\{\gamma\}}$ contained in $\widehat{N_G^A(N_\beta)}_{\{\gamma\}}$.*

Proof. The equality from assertion a) follows by using the epimorphism

$$s_\gamma : A_P^N \to A(P_\gamma)_P^{N_N(P_\gamma)}$$

of $N_G(P_\gamma)$-algebras and [5, Proposition 3.23], since $\bar\beta$ corresponds uniquely to β. For the proof of b), we note that the proof of the first equality of the statement is similar to that of a). The second equality is a consequence of the first equality. Next, for any $t \in N_N(P_\gamma)_i$ we have

$$A_\beta^P = (A_\beta \otimes t)^P.$$

This implies that any element $a' \in \phi((A_\gamma^*)^P) \subseteq (A_\beta^P)^*$ verifies

$$a' = a_1' \otimes t \in (A_\beta^P)^*,$$

where a_1' is a unit in A_β. Then, for a suitable $a \in A_\gamma^*$, we have

$$a_1' = a't^{-1} = at^{-1} + j - i \in \phi(A_\gamma^*).$$

By our choices of a, a' and a_1' we obtain $y^{t^{-1}} = y^{a_1'} = y^{at^{-1}}$, for all $y \in P$. Proposition 3.4 implies that $N_N(P_\gamma)_i$ is contained in $\widehat{N_G^A(N_\beta)}_{\{\gamma\}}$, since any element of N determines an A-fusion of N_β. We also have

$$N_N(P_\gamma)_i = N_N(P_\gamma) \cap \widehat{N_G(N_\beta)}_{\{\gamma\}} = N_N(P_\gamma) \cap \widehat{N_G(P_\gamma)}_{\{\bar\beta\}}.$$

Since $N_N(P_\gamma)$ is normal in $\widehat{N_G(N_\beta)}$ and in $((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar\beta\}})_{\bar j}$, the statement follows. □

3.10. We denote by $\bar G[\bar\beta]_{\{\gamma\}}$ the normal subgroup of $\bar N_G(P_\gamma)_{\{\bar\beta\}}$ that is isomorphic to $\bar G[\beta]_{\{\gamma\}}$ and by $\widehat{N_G^{A(P_\gamma)}}(P_\gamma)_{\{\bar\beta\}}$ the inverse image of $\bar G[\bar\beta]_{\{\gamma\}}$ in the infinite group of (ii), i.e. $((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar\beta\}})_{\bar j}$.

Remark 3.11. If $\hat x$ lifts an element of $\bar G[\bar\beta]_{\{\gamma\}}$ then, considering in the proof of Proposition 3.3 A in place of A_β , we obtain $(A \otimes \hat x)^P \cdot m_\gamma = m_\gamma \cdot (A \otimes \hat x)^P$. Then we set

$$\hat A(P_\gamma) := \bigoplus_{\hat x} \left((A \otimes \hat x)^P / (m_\gamma \cdot (A \otimes \hat x)^P)\right),$$

where $\hat x$ lifts a set of representatives of $\bar G[\bar\beta]_{\{\gamma\}}$. We denote

$$\overline{(A \otimes \hat x)^P} := (A \otimes \hat x)^P / (m_\gamma \cdot (A \otimes \hat x)^P),$$

for any lifting $\hat x$, and, using Lemma 3.9 and Proposition 3.3, we determine the strongly $\bar G[\bar\beta]_{\{\gamma\}}$-graded $N_N(P_\gamma)$-algebra

$$\hat A(P_\gamma)_{\bar\beta} := \bar j \hat A(P_\gamma) \bar j = (\hat A_\beta^\gamma)^P / \widehat{m}_\gamma = \bigoplus_{\hat x} \overline{(A_\beta \otimes \hat x)^P}.$$

Lemma 3.12. *The map*

$$\widehat{s}_\gamma : (\hat A_\beta^\gamma)^P \to \hat A(P_\gamma)_{\bar\beta},$$

sending $a \otimes \hat x$ to $\overline{a \otimes \hat x} := a \otimes \hat x + m_\gamma \cdot (A_\beta \otimes \hat x)^P$, is an epimorphism of $\bar G[\bar\beta]_{\{\gamma\}} \simeq \bar G[\beta]_{\{\gamma\}}$-strongly graded $N_N(P_\gamma)$-algebras.

Remark 3.13. We obtain the following characterization of $\bar{G}[\bar{\beta}]_{\{\gamma\}}$. Explicitly, it consists of elements $\bar{x} \in \bar{N}_G(P_\gamma)_{\bar{\beta}}$ such that x determines A-fusion of P_γ and for any lifting $\hat{x}$ we obtain

$$(\overline{(A_\beta \otimes \hat{x})^P})^{N_N(P_\gamma)} \cdot (\overline{(A_\beta \otimes \hat{x}^{-1})^P})^{N_N(P_\gamma)} = A(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}.$$

Further we construct a morphism of groups between $\widehat{N_G(N_\beta)}_{\{\gamma\}}$ and $\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$. Explicitly, if $\hat{n} := n \otimes z \in \widehat{N_G(N_\beta)}_{\{\gamma\}}$, where $z \in N_G(P_\gamma)_{\{\beta\}}$, then the image of $\hat{n}$ is $\widehat{s_\gamma(n)} := s_\gamma(n) \otimes z \in \widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$. So that we obtain a well-defined morphism of groups:

$$\theta : \widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i \to \widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}/N_N(P_\gamma)_{\bar{i}}.$$

3.14. By [5, Lemma 6.15], 3.13 and 3.12 and the above remark, the restriction

$$\widehat{s}_\gamma : (\hat{A}_\beta^\gamma)^N \to \hat{A}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}$$

is an epimorphism of $\widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i$-algebras, via the restriction determined by θ. By the definition of the action on $\overline{(A_\beta \otimes \hat{x})^P}$, this morphism verifies

$$\widehat{s}_\gamma((a \otimes \hat{x})^{\bar{\hat{n}}}) = (\widehat{s}_\gamma(a \otimes \hat{x}))^{\theta(\bar{\hat{n}})} := \overline{(n \otimes z)^{-1}(a \otimes \hat{x})(n \otimes z)},$$

for any $\bar{\hat{n}}$ in $\widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i$.

Corollary 3.15. *The group $\bar{G}[\bar{\beta}]_{\{\gamma\}}$ is invariant under the conjugation action determined by the elements belonging to the image of θ.*

Proof. For any $\hat{x}$ that lifts an element of $\bar{G}[\bar{\beta}]_\gamma$ we have

$$\widehat{s}_\gamma((A_\beta \otimes \hat{x})^N) = (\overline{(A_\beta \otimes \hat{x})^P})^{N_N(P_\gamma)}.$$

Using Lemma 3.6, Remark 3.13 and 3.13 the result follows. □

3.16. Denote

$$\hat{k}_1 := A(P_\gamma)_{\bar{\beta}}(N_N(P_\gamma)_{\bar{\beta}}) = A(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}/J(A(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}).$$

The twisted group algebra

$$\hat{\bar{A}}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)} := \hat{A}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}/J_{gr}(\hat{A}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}),$$

of $\hat{k}_1$ with $\bar{G}[\bar{\beta}]_{\{\gamma\}}$, corresponds uniquely to the extension

$$1 \to \hat{k}_1^* \to \widehat{\bar{N}_G^{A(P_\gamma)}(P_\gamma)}_{\{\bar{\beta}\}} \to \bar{G}[\bar{\beta}]_{\{\gamma\}} \to 1. \tag{2}$$

We call this the Clifford extension of the multiplicity algebra of γ. Note that $\hat{k}_1$ is a skew field having the center a finite extension of k. Moreover, we observe that $\hat{\bar{A}}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}$ is actually a crossed product of $\hat{k}_1$ with $\bar{G}[\bar{\beta}]_{\{\gamma\}}$, implying that it is a strongly $\bar{G}[\bar{\beta}]_{\{\gamma\}}$-graded algebra.

4. The restricted isomorphism of Clifford extensions

We summarize all of the above in the main result of the paper and we refer to [1, Theorem 4.1] for more details regarding the proof.

Theorem 4.1. *The following statements hold.*

(i) *The extensions* $(1')$ *and* (2) *are isomorphic.*

(ii) *The crossed products they correspond to are isomorphic as* $\widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i$-*algebras.*

Proof. The epimorphism $\widehat{s}_\gamma$ of $\bar{G}[\beta]_{\{\gamma\}} \simeq \bar{G}[\bar{\beta}]_{\{\gamma\}}$-strongly graded algebras determines the epimorphism

$$\widehat{\bar{s}}_\gamma : (\hat{\bar{A}}^\gamma_\beta)^N \to \hat{\bar{A}}(P_\gamma)^{N_N(P_\gamma)}_{\bar{\beta}},$$

since $\widehat{s}_\gamma(J((A^\gamma_\beta)^N)) \subseteq J(A(P_\gamma)^{N_N(P_\gamma)}_{\bar{\beta}})$. The map $\widehat{\bar{s}}_\gamma$ is also a morphism of $\bar{G}[\beta]_{\{\gamma\}}$-strongly graded algebras. [5, Proposition 3.23] gives $\hat{k} \simeq \hat{k}_1$, and the first assertion follows from [3, Proposition 2.12]. As for the second assertion we use Lemma 3.12, Remark 3.13 and Corollary 3.15. □

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