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Existence and localization of positive solutions to first order differential systems with nonlocal conditions

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Abstract. The purpose of the present work is to study the existence and the localization of positive solutions to nonlocal boundary value problems for first order differential systems. The localization is established by the vector version of Krasnosel'skiĭ's fixed point theorem in cones.

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1. Introduction

Nonlocal problems for different classes of differential equations and systems have been intensively studied in the literature (see, for example, [1], [2], [3], [9] for multi-point nonlocal conditions, and [13], [14] for nonlocal conditions given by Stieltjes integrals). One of the most common technique for the existence and localization of positive solutions to integral and differential equations is based on Krasnosel'skiĭ's fixed point theorem in cones (see, e.g. [4], [7], [8], [11] and [12]).

Motivated by the article of Li and Sun [6], in this paper, we study systems of first order equations with integral boundary conditions, using the vector version of Krasnosel'skiĭ's fixed point theorem in cones given by Precup [10]. This vectorial method allows the nonlinear terms of a system to have different behaviors both in components and variables. More exactly, in this paper we consider the following first order differential system with nonlocal boundary conditions given by linear functionals:

$$\begin{cases} u_1' = f_1(t, u_1, u_2) \\ u_2' = f_2(t, u_1, u_2) \\ u_1(0) - a_1 u_1(1) = g_1[u_1] \\ u_2(0) - a_2 u_2(1) = g_2[u_2] \end{cases} \tag{1.1}$$

where $f_1, f_2 \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}_+)$; $g_1, g_2 : C[0,1] \to \mathbb{R}$ are two linear functionals given by

$$g_i[u] = \int_0^1 u(s)\, d\gamma_i(s), \tag{1.2}$$

with $g_i[1] < 1$; $\gamma_i \in C^1[0,1]$ increasing and $0 < a_i < 1 - g_i[1]$ $(i = 1, 2)$.

We seek nonnegative solutions (u_1, u_2), $u_1 \geq 0$, $u_2 \geq 0$ on $[0,1]$.

1.1. The integral form of the nonlocal problem

In order to put (1.1) in an operator form, let us first consider the scalar problem:

$$\begin{cases} Lu := u' = h(t), & 0 \leq t \leq 1 \\ u(0) - au(1) = g[u] \end{cases} \tag{1.3}$$

where $h \in C[0,1]$; $g : C[0,1] \to \mathbb{R}$ is a linear functional given by

$$g[u] = \int_0^1 u(s)\, d\gamma(s), \tag{1.4}$$

with $g[1] < 1$; $\gamma \in C^1[0,1]$ increasing; $0 < a < 1 - g[1]$. We shall obtain the integral equation equivalent to the problem (1.3). To this end, we start with the differential equation, which by integration gives

$$u(t) = u(0) + \int_0^t h(s)\, ds. \tag{1.5}$$

Apply g to (1.5) and use its linearity to obtain

$$g[u] = u(0)g[1] + g\left[\int_0^t h(s)\, ds\right].$$

Notice that by $g\left[\int_0^t h(s)\, ds\right]$ we mean the value of functional g on the function $t \mapsto \int_0^t h(s)\, ds$. This together with the boundary condition in (1.3) yields

$$u(0) - au(1) = u(0)g[1] + g\left[\int_0^t h(s)\, ds\right],$$

and then

$$u(0) - u(0)g[1] = au(1) + g\left[\int_0^t h(s)\, ds\right].$$

On the other hand,

$$u(1) = u(0) + \int_0^1 h(s)\, ds.$$

Therefore

$$u(0) - u(0)g[1] = au(0) + a\int_0^1 h(s)\, ds + g\left[\int_0^t h(s)\, ds\right].$$

Hence

$$u(0) = \frac{1}{1 - g[1] - a}\left(g\left[\int_0^t h(s)\, ds\right] + a\int_0^1 h(s)\, ds\right).$$

If we denote $c := \dfrac{1}{1-g[1]-a}$ and we substitute into (1.5), we obtain

$$u(t) = cg\left[\int_0^t h(s)\,ds\right] + ca\int_0^1 h(s)\,ds + \int_0^t h(s)\,ds.$$

Next using the expresion (1.4) of g, we find

$$\begin{aligned}
u(t) &= c\int_0^1 \left(\int_0^s h(r)\,dr\right) d\gamma(s) + ca\int_0^1 h(s)\,ds + \int_0^t h(s)\,ds \\
&= c\int_0^1 \gamma^{'}(s)\int_0^s h(r)\,dr\,ds + ca\int_0^1 h(s)\,ds + \int_0^t h(s)\,ds \\
&= \int_0^t \left(c\gamma^{'}(s)\int_0^s h(r)\,dr + h(s)\right) ds + c\int_t^1 \gamma^{'}(s)\int_0^s h(r)\,dr\,ds \\
&\quad + ca\int_0^1 h(s)\,ds.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
u(t) &= c\gamma(s)\int_0^s h(r)\,dr\Big|_0^t - \int_0^t c\gamma(s)h(s)\,ds + c\gamma(s)\int_0^s h(r)\,dr\Big|_t^1 \\
&\quad - \int_t^1 c\gamma(s)h(s)\,ds + \int_0^t h(s)\,ds + ca\int_0^1 h(s)\,ds \\
&= c\gamma(1)\int_0^1 h(s)\,ds - \int_0^1 c\gamma(s)h(s)\,ds + \int_0^t h(s)\,ds + ca\int_0^1 h(s)\,ds \\
&= \int_0^1 c\left(\gamma(1)-\gamma(s)+a\right)h(s)\,ds + \int_0^t h(s)\,ds \\
&= \int_0^t [c(\gamma(1)-\gamma(s)+a)+1]h(s)\,ds + \int_t^1 c(\gamma(1)-\gamma(s)+a)h(s)\,ds. \qquad (1.6)
\end{aligned}$$

If now, to the nonlocal condition $u(0) - au(1) = g[u]$, we associate the Green function

$$G(t,s) = \begin{cases} c[\gamma(1)-\gamma(s)+a]+1 & \text{for } 0 \le s \le t \le 1 \\ c[\gamma(1)-\gamma(s)+a] & \text{for } 0 \le t < s \le 1, \end{cases} \qquad (1.7)$$

then (1.6) can be written as

$$u(t) = \int_0^1 G(t,s)h(s)\,ds.$$

Thus we have obtained the inverse of the operator L, $L^{-1} : C[0,1] \to C[0,1]$,

$$(L^{-1}h)(t) = \int_0^1 G(t,s)h(s)\,ds.$$

Based on this, the problem of nonnegative solutions of (1.1) is equivalent to the integral system:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G_1(t,s)f_1(s,u_1(s),u_2(s))\,ds \\ u_2(t) = \displaystyle\int_0^1 G_2(t,s)f_2(s,u_1(s),u_2(s))\,ds, \end{cases} \qquad (1.8)$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green functions corresponding to the two nonlocal conditions,

$$G_i(t,s) = \begin{cases} c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1 & \text{for } 0 \le s \le t \le 1 \\ c_i[\gamma_i(1) - \gamma_i(s) + a_i] & \text{for } 0 \le t < s \le 1, \end{cases}$$

where $c_i = \dfrac{1}{1 - g_i[1] - a_i}$ $(i = 1, 2)$.

The following properties are essential for the applicability of Krasnosel'skiĭ's technique:

1) $G_i(t,s) \le H_i(s)$, for all $t, s \in [0,1]$, where

$$H_i(s) = c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1 \ (i = 1, 2)$$

2) $\delta_i H_i(s) \le G_i(t,s)$ for all $t, s \in [0,1]$, where

$$\delta_i = \min_{s\in[0,1]} \frac{c_i[\gamma_i(1) - \gamma_i(s) + a_i]}{c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1} \ (i = 1, 2).$$

Let $N : C([0,1], \mathbb{R}^2_+) \to C([0,1], \mathbb{R}^2_+)$, $N = (N_1, N_2)$ be defined by

$$N_i(u_1, u_2)(t) = \int_0^1 G_i(t,s) f_i(s, u_1(s), u_2(s))\, ds \ (i = 1, 2).$$

The above properties of the Green functions imply that for any $t, t^* \in [0,1]$, one has:

$$\begin{aligned} N_i(u_1, u_2)(t) &= \int_0^1 G_i(t,s) f_i(s, u_1(s), u_2(s))\, ds \\ &\ge \delta_i \int_0^1 H_i(s) f_i(s, u_1(s), u_2(s))\, ds \\ &\ge \delta_i \int_0^1 G_i(t^*, s) f_i(s, u_1(s), u_2(s))\, ds = \delta_i N_i(u_1, u_2)(t^*). \end{aligned}$$

This yields the estimation from below

$$N_i(u_1, u_2)(t) \ge \delta_i |N_i(u_1, u_2)|_\infty \quad \text{for all } t \in [0,1] \ \ (i = 1, 2) \tag{1.9}$$

and any nonnegative functions $u_1, u_2 \in C[0,1]$.

Based on these estimations we define the cones:

$$K_i = \{u_i \in C[0,1] : u_i(t) \ge \delta_i |u_i|_\infty, \text{for all } t \in [0,1]\} (i = 1, 2), \tag{1.10}$$

and the product cone $K := K_1 \times K_2$ in $C([0,1], \mathbb{R}^2)$. Due to (1.9) we have the invariance property

$$N(K) \subset K.$$

Therefore, the problem of nonnegative solutions of (1.1) is equivalent to the fixed point problem

$$u = Nu,\ u \in K,$$

for the self-mapping N of K. Note that the continuity of f_1, f_2 implies the complete continuity of N.

Notice that (1.9) represents a weak Harnack type inequality for the nonnegative super solutions of the problem (1.1) (see [5]).

1.2. The vector version of Krasnosel'skiĭ's fixed point theorem in cones

The main tool of our paper is the following vector version of Krasnosel'skiĭ's fixed point theorem in cones given by Precup [10].

Theorem 1.1. *Let* $(X, |\cdot|)$ *be a normed linear space;* $K_1, K_2 \subset X$ *two cones;* $K := K_1 \times K_2$*;* $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$ *with* $0 < r < R$ *if* $0 < r_1 < R_1$ *and* $0 < r_2 < R_2$*;* $(K_i)_{r_i,R_i} = \{u_i \in K_i : r_i < |u_i| < R_i\}$*, for* $i = 1, 2$*;* $K_{r,R} := (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$ *and* $N : K_{r,R} \to K$, $N = (N_1, N_2)$ *a compact map. Assume that for each* $i \in \{1, 2\}$*, one of the following conditions is satisfied in* $K_{r,R}$*:*

(a) $N_i(u_1, u_2) \nleq u_i$ *if* $|u_i| = r_i$*, and* $N_i(u_1, u_2) \ngeq u_i$ *if* $|u_i| = R_i$*;*
(b) $N_i(u_1, u_2) \ngeq u_i$ *if* $|u_i| = r_i$*, and* $N_i(u_1, u_2) \nleq u_i$ *if* $|u_i| = R_i$*.*

Then N *has a fixed point* $u := (u_1, u_2)$ *in* K *with* $r_i < |u_i| < R_i$*, for* $i \in \{1, 2\}$*.*

Notice that the condition (a) means *compression*, while (b) means *expansion.* Therefore, in Theorem 1.1, the operators N_1, N_2 are both compressing, both expanding, or one compressing and the other one expanding.

2. Main results

2.1. Existence and localization

Using the notations from Section 1.1, we can state the main result of this paper.

Theorem 2.1. *Assume that there exist* α_i, $\beta_i > 0$ *with* $\alpha_i \neq \beta_i$, $i = 1, 2$*, such that*

$$\begin{aligned} A_1\lambda_1 > \alpha_1, \quad B_1\Lambda_1 < \beta_1, \\ A_2\lambda_2 > \alpha_2, \quad B_2\Lambda_2 < \beta_2, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} A_i &= \int_0^1 G_i(t^*, s)\, ds, \text{ for a chosen point } t^* \in [0, 1], \\ B_i &= \max_{0\le t\le 1} \int_0^1 G_i(t, s)\, ds, \\ \lambda_1 &= \min\{f_1(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1\alpha_1 \le u_1 \le \alpha_1,\ \delta_2 r_2 \le u_2 \le R_2\}, \\ \lambda_2 &= \min\{f_2(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1 r_1 \le u_1 \le R_1,\ \delta_2\alpha_2 \le u_2 \le \alpha_2\}, \\ \Lambda_1 &= \max\{f_1(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1\beta_1 \le u_1 \le \beta_1,\ \delta_2 r_2 \le u_2 \le R_2\}, \\ \Lambda_2 &= \max\{f_2(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1 r_1 \le u_1 \le R_1,\ \delta_2\beta_2 \le u_2 \le \beta_2\}, \end{aligned}$$

and $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1, 2)$*. Then* (1.1) *has at least one positive solution* $u = (u_1, u_2)$ *with* $r_i < |u_i|_\infty < R_i$ $(i = 1, 2)$*.*

Proof. We shall apply Theorem 1.1, with $X = C[0, 1]$, $|u| = \max_{0\le t\le 1} |u(t)|$ and K_1, K_2 given by (1.10).

If $u_i \in (K_i)_{r_i,R_i}$, then $r_i < |u_i|_\infty < R_i$ $(i = 1, 2)$. It follows from the definitions of K_i that

$$\delta_i r_i \le u_i(t) \le R_i \ (i = 1, 2)$$

for all $t \in [0, 1]$. Also, if we know for example that $|u_1|_\infty = \alpha_1$, then

$$\delta_1\alpha_1 \le u_1(t) \le \alpha_1.$$

We claim that for any $u_i \in (K_i)_{r_i,R_i}$ and $i \in \{1,2\}$, the following properties hold:

$$\begin{aligned} |u_i|_\infty = \alpha_i \quad &\text{implies} \quad N_i(u_1,u_2) \not\le u_i,\\ |u_i|_\infty = \beta_i \quad &\text{implies} \quad N_i(u_1,u_2) \not\ge u_i. \end{aligned} \tag{2.2}$$

Indeed, if $|u_1|_\infty = \alpha_1$ and we would have $N_1(u_1,u_2) \le u_1$, then for the chosen point t^* we obtain using (2.1):

$$\begin{aligned} &\alpha_1 \ge u_1(t^*) \ge N_1(u_1,u_2)(t^*) = \int_0^1 G_1(t^*,s) f_1(s,u_1(s),u_2(s))\,ds\\ &\ge A_1\lambda_1 > \alpha_1. \end{aligned}$$

This yields the contradiction $\alpha_1 > \alpha_1$. Now, if $|u_1|_\infty = \beta_1$ and $N_1(u_1,u_2) \ge u_1$, then for some $t' \in [0,1]$ with $|u_1|_\infty = u_1(t')$ we have

$$\begin{aligned} &\beta_1 = u_1(t') \le N_1(u_1,u_2)(t') = \int_0^1 G_1(t',s) f_1(s,u_1(s),u_2(s))\,ds\\ &\le B_1\Lambda_1 < \beta_1 \end{aligned}$$

whence we deduce that $\beta_1 < \beta_1$, a contradiction. Hence (2.2) holds for $i = 1$. Similary, (2.2) is true for $i = 2$. □

In particular, if f_1 and f_2 do not depend on t, i.e., $f_1 = f_1(u_1,u_2)$ and $f_2 = f_2(u_1,u_2)$, and f_1, f_2 have some monotonicity properties in u_1 and u_2, then we can specify the numbers λ_1, λ_2, Λ_1, Λ_2 and the conditions (2.1) are expressed by values of f_1, f_2 on only four points. Here are five cases from all the sixteen possible:

Case 1) If f_1, f_2 are nondecreasing in u_1 and u_2, then

$\lambda_1 = f_1(\delta_1\alpha_1, \delta_2 r_2)$, $\Lambda_1 = f_1(\beta_1, R_2)$, $\lambda_2 = f_2(\delta_1 r_1, \delta_2\alpha_2)$, $\Lambda_2 = f_2(R_1, \beta_2)$,

and (2.1) becomes

$$\begin{aligned} \frac{f_1(\delta_1\alpha_1,\delta_2 r_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1,R_2)}{\beta_1} < \frac{1}{B_1},\\ \frac{f_2(\delta_1 r_1,\delta_2\alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(R_1,\beta_2)}{\beta_2} < \frac{1}{B_2}. \end{aligned}$$

Case 2) If f_1 is nondecreasing in u_1 and u_2, while f_2 is nondecreasing in u_1 and nonincreasing in u_2, then

$\lambda_1 = f_1(\delta_1\alpha_1, \delta_2 r_2)$, $\Lambda_1 = f_1(\beta_1, R_2)$, $\lambda_2 = f_2(\delta_1 r_1, \alpha_2)$, $\Lambda_2 = f_2(R_1, \delta_2\beta_2)$,

and (2.1) reduces to

$$\begin{aligned} \frac{f_1(\delta_1\alpha_1,\delta_2 r_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1,R_2)}{\beta_1} < \frac{1}{B_1},\\ \frac{f_2(\delta_1 r_1,\alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(R_1,\delta_2\beta_2)}{\beta_2} < \frac{1}{B_2}. \end{aligned}$$

Case 3) If f_1 is nondecreasing in u_1 and nonincreasing in u_2, while f_2 is nonincreasing in u_1 and nondecreasing in u_2, then

$\lambda_1 = f_1(\delta_1\alpha_1, R_2)$, $\Lambda_1 = f_1(\beta_1, \delta_2 r_2)$, $\lambda_2 = f_2(R_1, \delta_2\alpha_2)$, $\Lambda_2 = f_2(\delta_1 r_1, \beta_2)$,

and (2.1) reads as

$$\frac{f_1(\delta_1\alpha_1, R_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1, \delta_2 r_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(R_1, \delta_2\alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(\delta_1 r_1, \beta_2)}{\beta_2} < \frac{1}{B_2}.$$

Case 4) If f_1 and f_2 are nondecreasing in u_1 and nonincreasing in u_2, then

$$\lambda_1 = f_1(\delta_1\alpha_1, R_2),\ \Lambda_1 = f_1(\beta_1, \delta_2 r_2),\ \lambda_2 = f_2(\delta_1 r_1, \alpha_2),\ \Lambda_2 = f_2(R_1, \delta_2\beta_2),$$

and (2.1) becomes

$$\frac{f_1(\delta_1\alpha_1, R_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1, \delta_2 r_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(\delta_1 r_1, \alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(R_1, \delta_2\beta_2)}{\beta_2} < \frac{1}{B_2}.$$

Case 5) If f_1 is nondecreasing in u_1 and u_2, while f_2 is nonincreasing in u_1 and u_2, then

$$\lambda_1 = f_1(\delta_1\alpha_1, \delta_2 r_2),\ \Lambda_1 = f_1(\beta_1, R_2),\ \lambda_2 = f_2(R_1, \alpha_2),\ \Lambda_2 = f_2(\delta_1 r_1, \delta_2\beta_2),$$

and (2.1) reduces to

$$\frac{f_1(\delta_1\alpha_1, \delta_2 r_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1, R_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(R_1, \alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(\delta_1 r_1, \delta_2\beta_2)}{\beta_2} < \frac{1}{B_2}.$$

2.2. Multiplicity

Theorem 2.1 guarantees the existence of solutions in an annular set. Clearly, if the assumptions of Theorem 2.1 are satisfied for several disjoint annular sets, then multiple solutions are obtained (see [11]).

Theorem 2.2. (A) *Let $(r^j)_{1\le j\le k}$, $(R^j)_{1\le j\le k}$ $(k \le \infty)$ be increasing finite or infinite sequence in $\mathbb{R}^2_+$, with $0 \le r^j < R^j$ and $R^j < r^{j+1}$ for all j. If the assumptions of Theorem* 2.1 *are satisfied for each couple (r^j, R^j), then* (1.1) *has k (respectively, when $k = \infty$, an infinite sequence of) distinct positive solutions.*

(B) *Let $(r^j)_{j\ge1}$, $(R^j)_{j\ge1}$ be decreasing infinite sequence with $0 < r^j < R^j$ and $R^j < r^{j+1}$ for all j. If the assumptions of Theorem* 2.1 *are satisfied for each couple (r^j, R^j), then* (1.1) *has an infinite sequence of distinct positive solutions.*

Proof. It is sufficient to see that

$$K_{r^j,R^j} \cap K_{r^{j+1},R^{j+1}} = \emptyset \text{ for all } j.$$

To prove this, let us consider two cases. First, if we assume that the sequences (r^j), (R^j) are increasing, then $K_{r^j,R^j} \subset \{u \in K : |u| < R^{j+1}\}$. Similary, if the sequences (r^j), (R^j) are decreasing, one has $K_{r^{j+1},R^{j+1}} \subset \{u \in K : |u| < r^j\}$. $\square$

2.3. Examples

We conclude by two examples illustrating Theorem 2.1 in the Cases 1) and 5).

Example 2.3. Let

$$f_i(u_1,u_2)=\frac{1}{15}\sqrt{u_1+u_2+1}\ \ for\ i=1,2,$$

$\gamma_1(t)=\frac{1}{2}t, \gamma_2(t)=\frac{3}{4}t$, $a_1=\frac{1}{4}$ and $a_2=\frac{1}{8}$. Then (1.1) becomes

$$\begin{cases} u_1'=\frac{1}{15}\sqrt{u_1+u_2+1} \\ u_2'=\frac{1}{15}\sqrt{u_1+u_2+1} \\ u_1(0)-\frac{1}{4}u_1(1)=\frac{1}{2}\int_0^1 u_1(t)\,dt \\ u_2(0)-\frac{1}{8}u_2(1)=\frac{3}{4}\int_0^1 u_2(t)\,dt, \end{cases} \tag{2.3}$$

or equivalently

$$\begin{cases} u_1(t)=\frac{1}{15}\int_0^1 G_1(t,s)\sqrt{u_1(s)+u_2(s)+1}\,ds \\ u_2(t)=\frac{1}{15}\int_0^1 G_2(t,s)\sqrt{u_1(s)+u_2(s)+1}\,ds, \end{cases} \tag{2.4}$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green functions

$$G_1(t,s)=\begin{cases} 6-4s \text{ for } 0\le s\le t\le 1 \\ 5-4s \text{ for } 0\le t<s\le 1, \end{cases}$$

$$G_2(t,s)=\begin{cases} 10-8s \text{ for } 0\le s\le t\le 1 \\ 9-8s \ \ \text{ for } 0\le t<s\le 1. \end{cases}$$

In this case, the constants $\delta_1,\delta_2>0$ are the following ones:

$$\delta_1=\delta_2=\frac{1}{2}=:\delta.$$

Now we have to determine A_i and B_i for $i\in\{1,2\}$. We have

$$A_1=\int_0^1 G_1(t^*,s)\,ds=\int_0^{t^*}(6-4s)\,ds+\int_{t^*}^1(5-4s)\,ds=t^*+3.$$

If we choose $t^*=0$, then $A_1=3$. Also

$$A_2=\int_0^1 G_2(t^*,s)\,ds=\int_0^{t^*}(10-8s)\,ds+\int_{t^*}^1(9-8s)\,ds=t^*+5,$$

and if we choose $t^*=0$, then $A_2=5$. In addition

$$B_1=\max_{0\le t\le 1}\int_0^1 G_1(t,s)\,ds=4,\quad B_2=\max_{0\le t\le 1}\int_0^1 G_2(t,s)\,ds=6.$$

In this case $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ are both nondecreasing in u_1 and u_2 for $u_1,u_2 \in \mathbb{R}_+$, so we are in Case 1). We choose $\alpha_1=\alpha_2=:\alpha$, $\beta_1=\beta_2=:\beta$, with $\alpha<\beta$, then $r_1=r_2=\alpha$, $R_1=R_2=\beta$ and $\lambda_1=f_1(\delta\alpha,\delta\alpha)$, $\Lambda_1=f_1(\beta,\beta)$, $\lambda_2=f_2(\delta\alpha,\delta\alpha)$, $\Lambda_2=f_2(\beta,\beta)$. The values of α and β will be precised in what follows. Since

$$\lim_{x\to\infty}\frac{f_i(x,x)}{x}=0 \quad\text{and}\quad \lim_{x\to 0}\frac{f_i(x,x)}{x}=\infty,$$

we may find α small enough and β large enough such that the conditions

$$\frac{f_i(\delta\alpha,\delta\alpha)}{\delta\alpha}>\frac{1}{\delta A_i},\quad \frac{f_i(\beta,\beta)}{\beta}<\frac{1}{B_i}\quad (i=1,2)$$

are satisfied. For instance, we can choose $\alpha=0,2$ and $\beta=0,7$.

Hence the following result holds.

Proposition 2.4. *The system (2.3) has at least one positive solution $u=(u_1,u_2)$ with*

$$0,2<|u_i|_\infty<0,7\ (i=1,2).$$

Example 2.5. Let $f_1(u_1,u_2)=\frac{1}{15}\sqrt{u_1+u_2+1}$, $f_2(u_1,u_2)=\frac{1}{(2+u_1^2)(4+u_2^2)}$, $\gamma_1(t)=\frac{1}{2}t$, $\gamma_2(t)=\frac{3}{4}t$, $a_1=\frac{1}{4}$ and $a_2=\frac{1}{8}$. Then (1.1) becomes

$$\begin{cases} u_1'=\dfrac{1}{15}\sqrt{u_1+u_2+1}\\ u_2'=\dfrac{1}{(2+u_1^2)(4+u_2^2)}\\ u_1(0)-\dfrac{1}{4}u_1(1)=\dfrac{1}{2}\displaystyle\int_0^1 u_1(t)\,dt\\ u_2(0)-\dfrac{1}{8}u_2(1)=\dfrac{3}{4}\displaystyle\int_0^1 u_2(t)\,dt, \end{cases}\tag{2.5}$$

or equivalently

$$\begin{cases} u_1(t)=\dfrac{1}{15}\displaystyle\int_0^1 G_1(t,s)\sqrt{u_1(s)+u_2(s)+1}\,ds\\ u_2(t)=\displaystyle\int_0^1 G_2(t,s)\frac{1}{(2+u_1(s)^2)(4+u_2(s)^2)}\,ds. \end{cases}\tag{2.6}$$

The Green functions $G_i(t,s)$ and the values of δ_i, A_i, B_i $(i=1,2)$ are the same from the Example 2.3. In this case $f_1(u_1,u_2)$ is nondecreasing in u_1 and u_2, while $f_2(u_1,u_2)$ is nonincreasing in u_1 and u_2, for $u_1,u_2\in\mathbb{R}_+$, so now we are in Case 5). We choose $\alpha_1=\alpha_2=:\alpha$, $\beta_1=\beta_2=:\beta$, with $\alpha<\beta$. Then $r_1=r_2=\alpha$, $R_1=R_2=\beta$ and $\lambda_1=f_1(\delta\alpha,\delta\alpha)$, $\Lambda_1=f_1(\beta,\beta)$, $\lambda_2=f_2(\beta,\alpha)$, $\Lambda_2=f_2(\delta\alpha,\delta\beta)$, where α and β will be precised in what follows. Since

$$\lim_{y\to\infty}\frac{f_1(y,y)}{y}=0 \quad\text{and}\quad \lim_{y\to\infty}\frac{f_2(x,y)}{y}=0,$$

uniformly in $x\geq 0$, we may find $\beta>0$ large enough such that

$$\frac{f_1(\beta,\beta)}{\beta}<\frac{1}{B_1},\quad \frac{f_2(\delta\alpha,\delta\beta)}{\delta\beta}<\frac{1}{\delta B_2}.$$

And since

$$\lim_{x\to 0}\frac{f_1(x,x)}{x}=\infty \quad \text{and} \quad \lim_{x\to 0}\frac{f_2(y,x)}{x}=0,$$

with β fixed as above, we choose α small enough such that

$$\frac{f_1(\delta\alpha,\delta\alpha)}{\delta\alpha}>\frac{1}{\delta A_1},\quad \frac{f_2(\beta,\alpha)}{\alpha}>\frac{1}{A_2}.$$

For example, we can choose $\beta=0,9$ and $\alpha=0,2$.

Hence the following result holds.

Proposition 2.6. *The system* (2.5) *has at least one positive solution* $u=(u_1,u_2)$ *with*

$$0,2<|u_i|_\infty<0,9\ (i=1,2).$$

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References

[1] Boucherif, A., *Differential equations with nonlocal boundary conditions*, Nonlinear Anal., **47**(2001), 2419-2430.

[2] Boucherif, A., Precup, R., *On nonlocal initial value problem for first order differential equations*, Fixed Point Theory, **4**(2003), no. 2, 205-212.

[3] Byszewski, L., *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162**(1991), 494-505.

[4] Erbe, L.H., Hu, S., Wang, H., *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl., **184**(1994), 640-648.

[5] Kassmann, M., *Harnack inequalities: an introduction*, Boundary Value Problems 2007, Article ID 81415, 21 pages, doi:10.1155/2007/81415.

[6] Li, Y., Sun, L., *Infinite many positive solutions for nonlinear first-order BVPS with integral boundary conditions on time scales*, Topol. Methods Nonlinear Anal., **41**(2013), 305-321.

[7] Lian, W.-C., Wong, F.-H., Yeh, C.-C., *On the existence of positive solutions of nonlinear second order differential equations*, Proc. Amer. Math. Soc., **124**(1996), 1117-1126.

[8] Meehan, M., O'Regan, D., *Multiple nonnegative solutions of nonlinear integral equations on compact and semi-infinite intervals*, Appl. Anal., **74**(2000), 413-427.

[9] Nica, O., Precup, R., *On the nonlocal initial value problem for first order differential systems*, Stud. Univ. Babeş-Bolyai Math., **56**(2011), no. 3, 113-125.

[10] Precup, R., *A vector version of Krasnosel'skiĭ's fixed point theorem in cones and positive periodic solutions on nonlinear systems*, J. Fixed Point Theory Appl., **2**(2007), 141-151.

[11] Precup, R., *Abstract week Harnack inequality, multiple fixed points and p-Laplace equations*, J. Fixed Point Theory Appl., **12**(2012), 193-206.

[12] Precup, R., *Moser-Harnack inequality, Krasnosel'skiĭ type fixed point theorems in cones and elliptic problems*, Topol. Methods Nonlinear Anal., **40**(2012), 301-313.

[13] Precup, R., Trif, D., *Multiple positive solutions of non-local initial value problems for first order differential systems*, Nonlinear Anal., **75**(2012), 5961-5970.

[14] Webb, J.R.L., Infante, G., *Non-local boundary value problems of arbitrary order*, J. Lond. Math. Soc., **79**(2008), 238-258.

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