Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 1, 123–128

Book reviews

Simeon Reich and Alexander J. Zaslavski, Genericity in Nonlinear Analysis, Developments in Mathematics, Vol. 34, Springer New York Heidelberg Dordrecht London, 2014, ISBN 978-1-4614-9532-1; ISBN 978-1-4614-9533-8 (eBook), xiii + 520 pp.

The book is concerned with generic (in the sense of Baire category) results for various problems in Nonlinear Analysis. As the authors best explain in the Preface, one considers a class of problems in some functional space equipped with a complete metric. It is known that for some elements in this functional space the corresponding problem possesses a solution (a solution with desirable properties) and for some elements such solutions do not exist. Under these circumstances it is natural to ask if a solution (a solution with desirable properties) exists for most (in the sense of Baire category) elements, meaning that this holds for a dense G_δ subset of the considered function space. In some cases, "most" is taken in the stronger sense of σ-porousity (a σ-porous set is of first Baire category and, in finite dimension, of Lebesgue measure zero too).

The classes of problems to which this general procedure is applied are: fixed point problems for both single- and set-valued mappings, infinite products of operators, best approximation, discrete and continuous descent methods for minimization in Banach spaces, and the structure of minimal energy configurations with rational numbers in the Aubry-Mather theory.

The first chapter of the book, *Introduction*, contains an overview of the principal results on fixed points and infinite products, which are then treated in detail in chapters 2. *Fixed Point Results and Convergence of Powers of Operators*, 3. *Contractive Mappings*, 6. *Infinite Products*, and 9. *Set-Valued Mappings*. Some results are presented in the general framework of a hyperbolic space, meaning a metric space (X, ρ) endowed with a family M of metric lines (isometric images of $\mathbb{R}^1$) such that every pair of points in X is joined by a unique metric line in M and the metric satisfies an inequality, expressing, intuitively, the fact that the length of a median in a triangle is less or equal than half the length of the base. The study of genericity in fixed point problems was initiated by G. Vidossich (1974) and F. De Blasi and J. Myjak (1976).

For a nonempty, bounded and closed subset K of a Banach space X one denotes by $\mathcal{A}$ the family of all nonexpansive mappings $A : K \to K$ (meaning that $\|Ax - Ay\| \leq \|x - y\|$, for all $x, y \in K$). Equipped with the metric $d(A, B) = \sup\{\|Ax - Bx\| : x \in K\}$ $\mathcal{A}$ is a complete metric space. A strict contraction is a γ-Lipschitz mapping with $0 \leq \gamma < 1$. A contractive mapping (in Rakotch's sense) is a mapping $A \in \mathcal{A}$ such that there exists a decreasing function $\phi^A : [0, \operatorname{diam}(K)] \to [0,1]$ such that $\|Ax - Ay\| \leq \phi^A(\|x-y\|)\,\|x-y\|$, for all $x, y \in K$. E. Rakotch (1962) proved that, under the above hypotheses, every contractive mapping in $\mathcal{A}$ has a unique fixed point x_A and that the sequence of iterates $(A^n x)$ converges to x_A uniformly on K. The authors show that there exists a G_δ dense subset $\mathcal{F}$ of $\mathcal{A}$ such that every $A \in \mathcal{F}$ is contractive. It is worth to notice that the set of all strict contractions is of first Baire category in $\mathcal{A}$, even if X is a Hilbert space. Also there exists a subset $\mathcal{F}$ of $\mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous and the conclusions of Rakotch's theorem hold for every $A \in \mathcal{F}$. Similar results, with respect to some Hausdorff-type metrics, are obtained for contractive set-valued mappings. The more delicate problem of non-expansive set-valued mappings is considered as well. Other generic results concern Mann-Ishikava iteration, stability of fixed points and the well-posedness of fixed point problems. The case of mappings which are non-expansive with respect to the Bergman metric (a topic of intense study in the recent years) is treated in Chapter 5, *Relatively Nonexpansive Operators with Respect to Bregman Distances.*

In Chapter 7, *Best Approximation*, one obtains generic and porousity results for generalized problems of best approximation in Banach spaces, or in the more general context of hyperbolic spaces. These results extend those obtained By S. B. Stechkin (1963), M. Edelstein (1968), Ka Sing Lau (1978), F. De Blasi and J. Myjak (1991,1998), Chong Li (2000), and others. The generalizations consist both in replacing the norm by a function f having some appropriate properties (convexity will do) and admitting that the set A, where the optimal points are checked, can vary. As a sample I do mention the following result: There exists a set $\mathcal{F}$ with σ-porous (with respect to some Hausdorf-type metric) complement in the space $S(X)$ of nonempty closed subsets of a complete hyperbolic space X such that the minimization problem is well posed for all points in X, excepting a subset of first Baire category (Theorem 7.5).

Generic and porousity results are obtained also for other classes of problems in chapters 4. *Dynamical Systems with Convex Lyapunov Functions*, 8. *Descent Methods*, and 10. *Minimal Configurations in the Aubry-Mather Theory.*

The book, based almost exclusively on the original results of the authors published in the last years, contains a lot of interesting and deep generic existence results for some classes of problems in nonlinear analysis. By bringing together results spread through various journals, it will be of great help for researchers in fixed point theory, optimization, best approximation and dynamical systems. Being carefully written, with complete proofs and illuminating examples, it can serve also as an introductory book to this areas of current research.

S. Cobzaş

Jürgen Appell, Józef Banaś and Nelson Merentes, Bounded Variation and Around, Series in Nonlinear Analysis and Applications, Vol. 17, xiii + 319 pp, Walter de Gruyter, Berlin - New York, 2014, ISBN: 978-3-11-026507-1, e-ISBN: 978-3-11-026511-8, ISSN: 0941-813X.

Functions of bounded variation were defined and studied first by Camille Jordan (1881) in order to extend the Dirichlet convergence criterium for Fourier series, who proved that these functions can be represented as differences of nondecreasing functions. Charles De la Vallée Poussin (1915) introduced the functions of bounded second variation and proved that these functions can be represented as differences of convex functions. T. Popoviciu (1934) generalized both these results by considering functions of bounded k-variation and proving that they can be written as differences of functions convex of order k, another class of functions defined and studied by T. Popoviciu (in *Introduction* his name is wrongly mentioned as Mihael T. Popoviciu – in item [253] the author is M. T. Popoviciu, but M. comes rather from Monsieur than from Mihael). Later on many extensions of this notion were considered, motivated mainly by their applications to Fourier series. We shall mention some of them which are treated in detail in this book.

In what follows f will be a function $f:[a,b]\to\mathbb{R}$ and P a partition $a=t_0<t_1<\cdots<t_m=b$ of the interval $[a,b]$.

F. Riesz (1910) considered the following variation $\operatorname{Var}_p^R(f,P)=\sum_{j=1}^m |f(t_j)-f(t_{j-1})|^p/(t_j-t_{j-1})^{p-1}$, and the total variation $\operatorname{Var}_p^R(f)=\sup_P \operatorname{Var}_p^R(f,P)$ where $p\geq 1$. He proved that a function f is of bounded Riesz variation iff it is absolutely continuous and $f'\in L^p([a,b])$. In this case $\operatorname{Var}_p^R(f)=\|f'\|_p^p$. He also used this class of functions to give representations for the duals of the spaces $L^p([a,b])$.

N. Wiener (1924) considered the total variation $\operatorname{Var}_p^W(f)$ as the supremum with respect to P of the variations $\operatorname{Var}_p^W(f,P)=\sum_{j=1}^m |f(t_j)-f(t_{j-1})|^p$. This was extended by L. Young (1934) to variations of the form $\operatorname{Var}_\phi^W(f,P)=\sum_{j=1}^m \phi(|f(t_j)-f(t_{j-1})|)$, where ϕ is a Young function. A similar extension to Riesz definition was given by Yu. T. Medvedev (1953). Daniel Waterman (1976) considered a more general notion of bounded variation using infinite sequences $\Lambda=(\lambda_n)$ of positive numbers with $\lambda_n\to 0$ and $\sum_n \lambda_n=\infty$, and infinite systems of nonoverlapping intervals $([a_n,b_n])$ in [a,b]: $\operatorname{Var}_\Lambda(f)=\sum_{j=1}^\infty \lambda_j|f(t_j)-f(t_{j-1})|$. Other generalizations were considered by M. Schramm (1982) (containing all the above mentioned notions of bounded variation, but difficult to handle, due to its technicality), B. Korenblum (1975), and others.

The aim of this book is to present a detailed and systematic account of all these notions of bounded variations, the properties of the corresponding spaces, and relations between various classes of functions with bounded variation. In general, the authors restrict the treatment to functions of one real variable, excepting the second chapter, *Classical BV-spaces*, where functions of several variables of bounded variations are considered as well. Applications are given to nonlinear composition operator (in Chapter 5) and to nonlinear superposition operator (in Chapter 6), as well as to convergence of Fourier series and to integral representations, via Riemann-Stieltjes-type integrals, of continuous linear functionals on various Banach function spaces.

Integrals of Riemann-Stieltjes-type, corresponding to various notions of bounded variation, are treated in detail in Chapter 4, *Riemann-Stieltjes integrals.*

The book is the first one that presents in a systematic and exhaustive way the many-faceted aspects of the notion of bounded variation and its applications. Some open problems are stated throughout the book, and each chapter ends with a set of exercises, completing the main text (the more difficult ones are marked by *). The book is very well organized with an index of symbols, a notion index, and a rich bibliography (329 items). The prerequisites are modest – some familiarity with real analysis, functional analysis and elements of operator theory – and so it can be used both for an introduction to the subject or as a reference text as well.

Tiberiu Trif

Lucio Boccardo and Gisella Croce, Elliptic Partial Differential Equations - Existence and Regularity of Distributional Solutions, Studies in Mathematics, Vol. 55, x + 192 pp, Walter de Gruyter, Berlin - New York, 2014, ISBN: 978-3-11-031540-0, e-ISBN: 978-3-11-031542-4, ISSN: 0179-0986.

The book is concerned with the existence and regularity of weak solutions to elliptic problems in divergence form (1) $-\mathrm{div}(a(x,u,\nabla u) = f$ in Ω and $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is open and bounded, $f \in H^{-1}(\Omega)$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is elliptic, that is $a(x,s,\xi) \cdot \xi \geq \alpha|\xi|^2$, $\xi \in \mathbb{R}^N$. The approach is based on methods from real and functional analysis. In order to make the book self-contained, many auxiliary results are proved with full details. Also, for reader's convenience, other results from real analysis, functional analysis, and Sobolev spaces, are collected in appendices at the end of some chapters, with reference to the recent book by H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer 2011.

The second chapter of the book (actually the first, because the Introduction is numbered as Chapter 1), *Some fixed point theorems*, contains Banach-Caccioppoli (called also Banach-Picard, or simply Banach, depending on the nationality of the author) contraction principle, Brouwer's fixed point theorem (Milnor's proof), and Schauder fixed point theorem. The third chapter, *Preliminaries of real analysis*, is concerned with some convergence results in L^p-spaces, other results on L^p-spaces being collected in an Appendix at the end of the chapter. The second part of this chapter contains a brief introduction to Marcinkiewicz spaces.

The study of elliptic equations starts in Chapter 4, *Linear and semilinear elliptic equations.* The existence results for this kind of equations are proved via Lax-Milgram and Stampacchia's theorems, whose full proofs are included. The Appendix to this chapter reviews some results in functional analysis (projections in Hilbert space and Riesz' theorem) and on Sobolev spaces.

The problems treated in Chapter 5, *Nonlinear elliptic equations*, are more difficult, due to the nonlinearity F in the equation $-\mathrm{div}(a(x,u,\nabla u) = F(x,u,\nabla u)$. The approach is based on some surjectivity results for pseudomonotone coercive operators on reflexive Banach spaces. The basic result proved here is the Leray-Lions existence theorem.

Chapter 6, *Summability of the solutions*, is concerned with some regularity results. One shows that the regularity of the solution depends on the regularity of the source f – if f belongs to a Lebesgue or Marcinkiewicz space, then the solution belongs to the same type of space (with modified exponents). This study continues in Chapter 7, *H^2 regularity for linear problems.* In Chapter 8, *Spectral analysis of linear operators,* one studies the eigenvalues of elliptic operators with applications to semilinear equations.

The last chapter of the first part of the book, 9, *Calculus of variations and Euler's equation*, is concerned with the existence of minimizers of weakly lsc integral functionals (De Giorgi's results), Euler's equation, and Ekeland's variational principle with applications.

The second part of the book is devoted to more specialized topics treated in chapters 10, *Natural growth problems*, 11, *Problems with low summable sources*, 12, *Uniqueness* (for both monotone and non-monotone elliptic operators), 13, *A problem with polynomial growth*, and 14, *A problem with degenerate coercivity.*

Based on undergraduate and Ph.D. courses, taught by the first author at La Sapienza University of Rome, this elegant book (dedicated to Bernardo Dacorogna for his 60th anniversary) presents, in an accessible but rigorous way, some basic results on elliptic partial differential equations. It can be used for undergraduate or graduate courses, or for introduction to this active area of investigation.

Radu Precup

Boris S. Mordukhovich and Nguyen Mau Nam, An Easy Path to Convex Analysis and Applications, Synthesis Lectures on Mathematics and Statistics, Vol. 6, No. 2, 218 pp., Morgan & Claypool, 2014, ISBN-10: 1627052372 ; ISBN-13: 978-1627052375.

The aim of the present book is to prepare the reader for the study of more advanced topics in nonsmooth and variational analysis. The authors have adopted a geometric approach, emphasizing the connections of normal and tangent cones with the subdifferentials of convex functions as well as their relevance for optimization problems. In this way they offer intuitive and more digestible models for the variety of cones considered in nonsmooth analysis. The accessibility is further stressed by the restriction to the framework of finite dimensional Euclidean space $\mathbb{R}^n$.

The first chapter of the book, *Convex sets and functions*, presents the basic notions and results in this area – convex hull, operations with convex sets, topological properties of convex sets, the algebraic interior, convex functions. Applications are given to the distance functions and to the optimal (or marginal) value function μ, defined by $\mu(x) = \inf\{\varphi(x,y) : y \in F(x)\}$, where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a set-valued mapping and $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function.

The development of the subdifferential calculus for convex functions, given in the second chapter, is based on a general separation theorem for convex subsets of $\mathbb{R}^n$ and on the normal cone $N(\bar{x};\Omega)$ to a convex set Ω. Calculus rules for normal cones are established and the continuity and differentiability properties of convex functions are studied. The authors consider two kinds of subdifferentials for a convex function $f : \Omega \to \overline{\mathbb{R}}$ – the singular subdifferential $\partial^\infty f(\bar{x}) = \{v \in \mathbb{R}^n : (v,0) \in N((\bar{x}, f(\bar{x})); \mathrm{epi} f)\}$

and the usual subdifferential $\partial f(\bar{x})$, defined as the set of subgradients of f at $\bar{x}$. As applications, the subdifferentials of the distance function and of the optimal value function are calculated. Some connections of the subdifferential of the optimal value function with the coderivative of the set-valued mapping F, a key tool in variational analysis, are established.

Chapter 3, *Remarkable consequences of convexity*, starts with the proof of the equivalence between the Fréchet and Gâteaux differentiability of convex functions and continues with Carathéodory's theorem on the convex hulls of subsets of $\mathbb{R}^n$, Radon's theorem, Helly's intersection theorem, and Farkas lemma on systems of linear inequalities. One introduces the tangent cone $T(\bar{x};\Omega)$ and one proves the duality relations $N(\bar{x};\Omega) = [T(\bar{x};\Omega)]^\circ$ and $T(\bar{x};\Omega) = [N(\bar{x};\Omega)]^\circ$, where $K^\circ = \{v \in \mathbb{R}^n : \langle v, x\rangle \le 0, \forall x \in K\}$ denotes the polar cone to a cone $K \subset \mathbb{R}^n$.

The last chapter, Ch. 4, *Applications to optimization and location problems*, contains some recent results of the authors on the Fermat-Torricelli and Sylvester problems – two problems with geometric flavor, where the methods of convex analysis proved to be very efficient for their solution.

The book is clearly and carefully written, with elegant and full proofs to almost all results. Some notions are accompanied by nicely drawn illustrative pictures, and the exercises at the end of each chapter help the reader to a broader and deeper understanding of the results from the main text.

The book contains an accessible, with a strong intuitive support but reasonably complete, introduction to some basic results in convex analysis in $\mathbb{R}^n$. It is recommended as a preliminary (or a companion) lecture to more advanced texts as, for instance, the monumental treatise of the first named author, B. S. Mordukhovich, *Variational Analaysis and Generalized Differentiation*, Vols. I and II, Springer, 2006 (up to now, Google Scholar counts 2015 citations of these volumes). It can be used also as a textbook for graduate or advanced undergraduate courses.

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