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Coupled fixed point theorems for mixed monotone operators and applications

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Abstract. In this paper we present some existence and uniqueness results for a coupled fixed point problem associated to a pair of singlevalued satisfying the mixed monotone property. We also provide an application to a first-order differential system with periodic boundary value conditions.

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1. Introduction

In the study of the fixed points for an operator, it is sometimes useful to consider a more general concept, namely coupled fixed point. The concept of coupled fixed point for nonlinear operators was introduced and studied by Opoitsev (see V.I. Opoitsev [7]-[9]) and then, in 1987, by D. Guo and V. Lakshmikantham (see [4]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations. Later, a new research direction for the theory of coupled fixed points in ordered metric space was initiated by T. Gnana Bhaskar and V. Lakshmikantham in [3] and by V. Lakshmikantham and L. Ćirić in [5]. T. Gnana Bhaskar and V. Lakshmikantham [3] introduced the notion of the mixed monotone property of a given operator. Furthermore, they proved some coupled fixed point theorems for operators which satify the mixed monotone property and presented as an application, the existence and uniqueness of a solution for a periodic boundary value problem. Their approach is based on some contractive type conditions on the operator.

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For more applications see for example A. C. M. Ran, M. C. B. Reurings [12], J. J. Nieto and R. R. López [6], W. Sintunavarat, P. Kumam, and Y. J. Cho [15], V. Lakshmikantham and L. Ćirić [5], C. Urs [17].

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(a) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

A set endowed with a vector-valued metric d is called generalized metric space. The notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, ..., m\}.$$

Notice that we will make an identification between row and column vectors in $\mathbb{R}^m$.

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see G. Allaire and S. M. Kaber [1], I. A. Rus [13], R. S. Varga [18]).

Theorem 1.1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalents:*

(i) A is convergent towards zero;

(ii) $A^n \to 0$ as $n \to \infty$;

(iii) The eigenvalues of A are in the open unit disc, i.e $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;

(iv) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + ... + A^n + ...; \tag{1.1}$$

(v) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;

(vi) $A^n q \to 0$ and $q A^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

In the second section of this paper, the aim is to present in the setting of an ordered metric space, a Gnana Bhaskar-Lakshmikantham type theorem for the coupled fixed point problem of a pair of singlevalued operators satisfying a generalized mixed monotone assumption. In the third section we study the existence and uniqueness of the solution to a periodic boundary value system, as an application to the coupled fixed point theorem.

2. Existence and uniqueness results for coupled fixed point

Let X be a nonempty set endowed with a partial order relation denoted by $\leq$. Then we denote

$$X_{\leq} := \{(x_1, x_2) \in X \times X \ : \ x_1 \leq x_2 \text{ or } x_2 \leq x_1\}.$$

If $f : X \to X$ is an operator then we denote the cartesian product of f with itself as follows:

$$f \times f : X \times X \to X \times X, \text{ given by } (f \times f)(x_1, x_2) := (f(x_1), f(x_2)).$$

Definition 2.1. *Let X be a nonempty set. Then $(X, d, \leq)$ is called an ordered generalized metric space if:*

(i) (X, d) is a generalized metric space in the sense of Perov;

(ii) $(X, \leq)$ is a partially ordered set;

The following result will be an important tool in our approach.

Theorem 2.2. *Let $(X, d, \leq)$ be an ordered generalized metric space and let $f : X \to X$ be an operator. We suppose that:*

(1) for each $(x, y) \notin X_{\leq}$ there exists $z(x, y) := z \in X$ such that $(x, z), (y, z) \in X_{\leq}$;

(2) $X_{\leq} \in I(f \times f)$;

(3) $f : (X, d) \to (X, d)$ is continuous;

(4) the metric d is complete;

(5) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\leq}$;

(6) there exists a matrix $A \in M_{mm}(\mathbb{R}_+)$ which converges to zero, such that

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for each } (x, y) \in X_{\leq}.$$

Then $f : (X, d) \to (X, d)$ is a Picard operator.

Proof. Let $x \in X$ be arbitrary. Since $(x_0, f(x_0)) \in X_{\leq}$, by (6) and (4), we get that there exists $x^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$. By (3) we get that $x^* \in Fix(f)$.

If $(x, x_0) \in X_{\leq}$, then by (2), we have that $(f^n(x), f^n(x_0)) \in X_{\leq}$, for each $n \in \mathbb{N}$. Thus, by (6), we get that $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$.

If $(x, x_0) \notin X_{\leq}$, then by (1), it follows that there exists $z(x, x_0) := z \in X_{\leq}$ such that $(x, z), (x_0, z) \in X_{\leq}$. By the fact that $(x_0, z) \in X_{\leq}$, as before, we get that $(f^n(z))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$. This together with the fact that $(x, z) \in X_{\leq}$ implies that $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$.

Finally, the uniqueness of the fixed point follows by the contraction condition (6) using again the assumption (1). □

Remark 2.3. The conclusion of the above theorem holds if instead the hypothesis (2) we put:

(2') $f : (X, \leq) \to (X, \leq)$ is monotone increasing

or

(2") $f : (X, \leq) \to (X, \leq)$ is monotone decreasing.

Of course, it is easy to remark that assertion (2) in Theorem 2.2 is more general. For example, if we consider the ordered metric space $(\mathbb{R}^2, d, \leq)$, then $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x_1, x_2) := (g(x_1, x_2), g(x_1, x_2))$ satisfies (2), for any $g : \mathbb{R}^2 \to \mathbb{R}$.

Remark 2.4. Condition (5) from the above theorem is equivalent with:

$$(5') \quad f \text{ has a lower or an upper fixed point in } X.$$

Remark 2.5. For some similar results see Theorem 4.2 and Theorem 4.7 in A. Petruşel, I.A. Rus [10].

We will apply the above result for the coupled fixed point problem generated by two operators.

Let X be a nonempty set endowed with a partial order relation denoted by $\leq$. If we consider $z := (x, y), w := (u, v)$ two arbitrary elements of $Z := X \times X$, then, by definition

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v).$$

Notice that $\preceq$ is a partial order relation on Z.

We denote

$$Z_{\preceq} = \{(z, w) := ((x, y), (u, v)) \in Z \times Z \ : \ z \preceq w \text{ or } w \preceq z\}.$$

Let $T : Z \to Z$ be an operator defined by

$$T(x, y) := \begin{pmatrix} T_1(x, y) \\ T_2(x, y) \end{pmatrix} = (T_1(x, y), T_2(x, y)) \tag{2.1}$$

The cartesian product of T and T will be denoted by $T \times T$ and it is defined in the following way

$$T \times T : Z \times Z \to Z \times Z, \quad (T \times T)(z, w) := (T(z), T(w)).$$

We recall the following existence and uniqueness theorem for the coupled fixed point of a pair of singlevalued operators which satisfy the mixed monotone property (see A. Petrusel, G. Petrusel, and C. Urs [11]).

Theorem 2.6. *Let $(X, d, \leq)$ be an ordered and complete metric space and let $T_1, T_2 : X \times X \to X$ be two operators. We suppose:*

(i) for each $z = (x, y), w = (u, v) \in X \times X$ which are not comparable with respect to the partial ordering $\preceq$ on $X \times X$, there exists $t := (t_1, t_2) \in X \times X$ (which may depend on (x, y) and (u, v)) such that t is comparable (with respect to the partial ordering $\preceq$) with both z and w, i.e.,

$$((x \geq t_1 \text{ and } y \leq t_2) \text{ or } (x \leq t_1 \text{ and } y \geq t_2) \text{ and}$$
$$((u \geq t_1 \text{ and } v \leq t_2) \text{ or } (u \leq t_1 \text{ and } v \geq t_2));$$

(ii) for all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$ we have

$$\begin{cases} T_1(x, y) \geq T_1(u, v) \\ T_2(x, y) \leq T_2(u, v) \end{cases} \text{or} \begin{cases} T_1(u, v) \geq T_1(x, y) \\ T_2(u, v) \leq T_2(x, y) \end{cases}$$

(iii) $T_1, T_2 : X \times X \to X$ are continuous;

(iv) there exists $z_0 := (z_0^1, z_0^2) \in X \times X$ such that

$$\begin{cases} z_0^1 \geq T_1(z_0^1, z_0^2) \\ z_0^2 \leq T_2(z_0^1, z_0^2) \end{cases} \text{or} \begin{cases} T_1(z_0^1, z_0^2) \geq z_0^1 \\ T_2(z_0^1, z_0^2) \leq z_0^2 \end{cases}$$

(v) there exists a matrix $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in M_2(\mathbb{R}_+)$ *convergent toward zero such that*

$$\begin{aligned} d(T_1(x,y),T_1(u,v)) &\le k_1 d(x,u) + k_2 d(y,v) \\ d(T_2(x,y),T_2(u,v)) &\le k_3 d(x,u) + k_4 d(y,v) \end{aligned}$$

for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$.

Then there exists a unique element $(x^*, y^*) \in X \times X$ *such that*

$$x^* = T_1(x^*, y^*) \text{ and } y^* = T_2(x^*, y^*)$$

and the sequence of the succesive aproximations $(T_1^n(w_0^1, w_0^2), T_2^n(w_0^1, w_0^2))$ *converges to* (x^*, y^*) *as* $n \to \infty$, *for all* $w_0 = (w_0^1, w_0^2) \in X \times X$.

3. An application to periodic boundary value system

We study now the existence and uniqueness of the solution to a periodic boundary value system, as an application to coupled fixed point Theorem 2.6 for mixed monotone type singlevalued operators in the framework of partially ordered metric space.

In a similar context M. D. Rus [14] investigated the fixed point problem for a system of multivariate operators which are coordinate-wise uniformly monotone, in the setting of quasi-ordered sets. As an application, a new abstract multidimensional fixed point problem was studied. Additionally an application to a first-order differential system with periodic boundary value conditions was presented (see M. D. Rus [14]). V. Berinde and M. Borcut studied in [2] also the existence and uniqueness of solutions to a tripled fixed point problem.

We denote the partial order relation by $\preceq$ on $C(I) \times C(I)$. If we consider $z := (x, y)$ and $w := (u, w)$ two arbitrary elements of $C(I) \times C(I)$, then by definition

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v),$$

where $x \geq u$ means that $x(t) \geq u(t)$, for all $t \in I$.

We consider the first-order periodic boundary value system:

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ x(0) = x(T) \\ y(0) = y(T) \end{cases} \text{for all } t \in I := [0, T] \tag{3.1}$$

where $T > 0$ and $f_1, f_2 : I \times \mathbb{R}^2 \to \mathbb{R}$ under the assumptions:

(a1) f_1, f_2 are continuous;

(a2) there exist $\lambda > 0$ and $\mu_1, \mu_2, \mu_3, \mu_4 > 0$ such that

$$0 \leq f_1(t, x, y) - f_1(t, u, v) + \lambda(x - u) \leq \lambda[\mu_1(x - u) + \mu_2(y - v)]$$

$$-\lambda\left[\mu_3(x - u) + \mu_4(y - v)\right] \leq f_2(t, x, y) - f_2(t, u, v) + \lambda(y - v) \leq 0$$

for all $t \in I$ and $x, y, u, v \in \mathbb{R}$.

(a3) for each $z := (x, y), w := (u, w) \in C(I) \times C(I)$ which are not comparable with respect to the partial ordering $\preceq$ on $C(I) \times C(I)$ there exists $p := (p_1, p_2) \in$

$C(I)\times C(I)$ such that p is comparable (with respect to the partial ordering $\preceq$) with both z and w, i.e.

$$((x \geq p_1 \text{ and } y \leq p_2) \text{ or } (x \leq p_1 \text{ and } y \leq p_2) \text{ and}$$
$$(u \geq p_1 \text{ and } v \leq p_2) \text{ or } (u \leq p_1 \text{ and } v \leq p_2)).$$

(a4) for all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) we have

$$\begin{cases} f_1(t,x,y) \geq f_1(t,u,v) \\ f_2(t,x,y) \leq f_1(t,u,v) \end{cases} \text{or} \begin{cases} f_1(t,u,v) \geq f_1(t,x,y) \\ f_2(t,u,v) \leq f_2(t,x,y) \end{cases}.$$

(a5) there exists $z_0 := (z_0^1, z_0^2) \in C(I)\times C(I)$ such that the following relations hold:

$$\text{(a5')} \begin{cases} z_0^1(t) \geq f_1(t, z_0^1(t), z_0^2(t)) \\ z_0^2(t) \geq f_2(t, z_0^1(t), z_0^2(t)) \end{cases} \text{or} \begin{cases} f_1(t, z_0^1(t), z_0^2(t)) \geq z_0^1(t) \\ f_2(t, z_0^1(t), z_0^2(t)) \leq z_0^2(t) \end{cases}$$

$$\text{(a5")}\ (1+\lambda)\int_0^T G_\lambda(t,s) z_0^1(s)ds \geq z_0^1(t)$$

$$(1+\lambda)\int_0^T G_\lambda(t,s) z_0^2(s)ds \leq z_0^2(t)$$

for all $t \in I$.

(a6) the matrix $S := \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ is convergent to zero.

Lemma 3.1. *Let $x \in C^1(I)$ be such that it satisfies the periodic boundary value problem*

$$\begin{cases} x'(t) = h(t) \\ x(0) = x(T) \end{cases} t \in I,$$

with $h \in C(I)$. Then for some $\lambda \neq 0$ the above problem is equivalent to

$$x(t) = \int_0^T G_\lambda(t,s)(h(s) + \lambda x(s))ds, \text{ for all } t \in I,$$

where

$$G_\lambda(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \leq s < t \leq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \leq t < s \leq T \end{cases}.$$

The problem (3.1) is equivalent to the coupled fixed point problem

$$\begin{cases} x = F_1(x,y) \\ y = F_2(x,y) \end{cases},$$

with $X = C(I)$ and $F_1, F_2 : X^2 \to X$ defined by

$$F_1(x,y)(t) = \int_0^T G_\lambda(t,s)\left[f_1(s,x(s),y(s)) + \lambda x(s)\right] ds$$
$$F_2(x,y)(t) = \int_0^T G_\lambda(t,s)\left[f_2(s,x(s),y(s)) + \lambda y(s)\right] ds$$

We consider the complete metric d induced by the sup-norm on X,

$$d(x,y) = \sup_{t\in I} |x(t) - y(t)|\,, \text{ for } x,y \in C(I)$$

For $x,y,u,v \in X$, we also denote $\widetilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}$.

Note that if $(x,y) \in X \times X$ is a coupled fixed point of F, then we have

$$x(t) = F_1(x,y)(t) \text{ and } y(t) = F_2(x,y)(t) \text{ for all } t \in I,$$

where $F(x,y)(t) := (F_1(x,y)(t), F_2(x,y)(t))$.

Our main result in this section is the following theorem regarding the existence and uniqueness of the solution to a periodic boundary value system.

Theorem 3.2. *Consider the problem (*3.1*) under the assumptions (a1)-(a6). Then there exists a unique solution (x^*,y^*) of the first-order boundary value problem* 3.1*.*

Proof. We verify the conditions of Theorem 2.6.

$$F_1(x,y)(t) - F_1(u,v)(t)$$

$$= \int_0^T G_\lambda(t,s)\left[f_1(s,x(s),y(s)) + \lambda x(s)\right] ds - \int_0^T G_\lambda(t,s)\left[f_1(s,u(s),v(s)) + \lambda u(s)\right] ds$$

$$= \int_0^T G_\lambda(t,s)[f_1(s,x(s),y(s)) - f_1(s,u(s),v(s)) + \lambda(x(s) - u(s))]ds$$

for all $t \in I$.

From the first condition in (a2) and the positivity of G_λ (for $\lambda > 0$) it follows that

$$F_1(x,y)(t) - F_1(u,v)(t) \geq 0.$$

We get that

$$F_1(x,y)(t) \geq F_1(u,v)(t).$$

In a similar way we obtain

$$F_2(x,y)(t) - F_2(u,v)(t)$$

$$= \int_0^T G_\lambda(t,s)\left[f_2(s,x(s),y(s)) + \lambda y(s)\right] ds - \int_0^T G_\lambda(t,s)\left[f_2(s,u(s),v(s)) + \lambda v(s)\right] ds$$

$$= \int_0^T G_\lambda(t,s)[f_2(s,x(s),y(s)) - f_2(s,u(s),v(s)) + \lambda(y(s) - v(s))]ds$$

From the second inequality in (a2) and the positivity of G_λ (for $\lambda > 0$) we have

$$F_2(x,y)(t) - F_2(u,v)(t) \leq 0.$$

Hence it follows that

$$F_2(x,y)(t) \leq F_2(u,v)(t).$$

So we get that for all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) we have

$$\begin{cases} F_1(x,y)(t) \geq F_1(u,v)(t) \\ F_2(x,y)(t) \leq F_2(u,v)(t) \end{cases} \text{or} \quad \begin{cases} F_1(u,v)(t) \geq F_1(x,y)(t) \\ F_2(u,v)(t) \leq F_2(x,y)(t) \end{cases}$$

Hence the second hypothesis of Theorem 2.6 is satisfied.

We know that

$$F_1(z_0^1, z_0^2)(t) = \int_0^T G_\lambda(t,s)[f_1(s, z_0^1(s), z_0^2(s)) + \lambda z_0^1(s)]ds,$$

and we have from condition (a5') that

$$f_1(t, z_0^1(t), z_0^2(t)) \geq z_0^1(t).$$

So we get that

$$\begin{aligned} F_1(z_0^1, z_0^2)(t) &\geq \int_0^T G_\lambda(t,s)[z_0^1(s) + \lambda z_0^1(s)]ds \\ &= (1+\lambda)\int_0^T G_\lambda(t,s) z_0^1(s) ds \geq z_0^1(t) \end{aligned}$$

for all $t \in I$.

Finally we obtain that

$$F_1(z_0^1, z_0^2)(t) \geq z_0^1(t), \text{ for all } t \in I.$$

$$F_2(z_0^1, z_0^2)(t) = \int_0^T G_\lambda(t,s)[f_2(s, z_0^1(s), z_0^2(s)) + \lambda z_0^2(s)]ds$$

From condition (a5') we know that

$$f_2(t, z_0^1(t), z_0^2(t)) \leq z_0^2(t).$$

It follows that

$$\begin{aligned} F_2(z_0^1, z_0^2)(t) &\leq \int_0^T G_\lambda(t,s)[z_0^2(s) + \lambda z_0^2(s)]ds \\ &= (1+\lambda)\int_0^T G_\lambda(t,s) z_0^2(s) ds \leq z_0^2(t). \end{aligned}$$

Hence we have that

$$F_2(z_0^1, z_0^2)(t) \leq z_0^2(t).$$

We conclude that the fourth hypothesis of Theorem 2.6 is satisfied.

$$|F_1(x,y)(t) - F_1(u,v)(t)|$$

$$= \left| \int_0^T G_\lambda(t,s)[f_1(s, x(s), y(s)) + \lambda x(s)]ds - \int_0^T G_\lambda(t, s[f_1(s, u(s), v(s)) + \lambda u(s)]ds \right|$$

$$= \left| \int_0^T G_\lambda(t,s)[f_1(s, x(s), y(s)) - f_1(s, u(s), v(s)) + \lambda x(s) - \lambda u(s)]ds \right|$$

$$\leq \lambda \int_0^T G_\lambda(t,s)(|\mu_1(x(s) - u(s))| + |\mu_2(y(s) - v(s))|)ds$$

$$\leq \mu_1 d(x,u) + \mu_2 d(y,v).$$

Taking the $\sup_{t \in I}$ in the above relation we have

$$d(F_1(x,y), F_1(u,v)) \leq \mu_1 d(x,u) + \mu_2 d(y,v).$$

In a similar way we get that

$$d(F_2(x,y),F_2(u,v)) \leq \mu_3 d(x,u) + \mu_4 d(y,v).$$

Then we have

$$\begin{pmatrix} d(F_1(x,y),F_1(u,v)) \\ d(F_2(x,y),F_2(u,v)) \end{pmatrix} \leq \begin{pmatrix} \mu_1 d(x,u) + \mu_2 d(y,v) \\ \mu_3 d(x,u) + \mu_4 d(y,v) \end{pmatrix}$$
$$= \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix} = S \cdot \widetilde{d}((x,y),(u,v)),$$

where S is a matrix convergent to zero.

Since all the hypotheses of Theorem 2.6 are satisfied we get that the periodic boundary value problem (3.1) has a unique solution on $C(I) \times C(I)$. □

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