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Some fixed point theorems on cartesian product in terms of vectorial measures of noncompactness

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Abstract. In this paper we study a system of operatorial equations in terms of some vectorial measures of noncompactness. The basic tools are the cartesian hull of a subset of a cartesian product and some classical fixed point principle.

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1. Introduction

Let X_i, $i = \overline{1,m}$, be some nonempty sets, $X := \prod_{i=1}^{m} X_i$ and $f : X \to X$ be an operator. In this case the fixed point equation

$$x = f(x),$$

where $x = (x_1, \ldots, x_m)$ and $f = (f_1, \ldots, f_m)$ takes the following form

$$\begin{cases} x_1 = f_1(x_1, \ldots, x_m) \\ \vdots \\ x_m = f_m(x_1, \ldots, x_m) \end{cases}$$

In this paper we shall study the above system of operatorial equation in the case when X_i, $i = \overline{1,m}$, are metric spaces. In order to do this, we introduce the cartesian hull and vectorial measure of noncompactness.

2. Preliminaries

Let (X, d) be a metric space. In this paper we shall use the following notations:
$\mathcal{P}(X) = \{Y \mid Y \subset X\}$
$P(X) = \{Y \subset X \mid Y \text{ is nonempty}\}$, $P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}$,
$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$, $P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$,

$P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\}$.
If X is a Banach space then $P_{cv}(X) := \{Y \in P(X) |\ Y \text{ is convex}\}$
Let $f : X \to X$ is an operator. Then, we denote by $F_f := \{x \in X |\ x = f(x)\}$ the fixed point set of the operator f.

Definition 2.1. *A matrix $S \in \mathbb{R}_+^{m\times m}$ is called a matrix convergent to zero iff $S^k \to 0$ as $k \to +\infty$.*

Theorem 2.2. *(see [2], [16], [18], [20], [23]) Let $S \in \mathbb{R}_+^{m\times m}$. The following statements are equivalent:*

(i) *S is a matrix convergent to zero;*
(ii) *$S^k x \to 0$ as $k \to +\infty$, $\forall x \in \mathbb{R}^m$;*
(iii) *$I_m - S$ is non-singular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \dots$$

(iv) *$I_m - S$ is non-singular and $(I_m - S)^{-1}$ has nonnegative elements;*
(v) *$\lambda \in \mathbb{C}$, $\det(S - \lambda I_m) = 0$ imply $|\lambda| < 1$;*
(vi) *there exists at least one subordinate matrix norm such that $\|S\| < 1$.*

The matrices convergent to zero were used by A. I. Perov [15] to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^m$. For fixed point principle in such spaces see [16], [20], [22], [23].

3. Closure operators. Cartesian hull of a subset of a cartesian product

Let X be a nonempty set. By definition an operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator if:

(i) $Y \subset \eta(Y)$, $\forall Y \in \mathcal{P}(X)$;
(ii) $Y, Z \in \mathcal{P}(X)$, $Y \subset Z \Longrightarrow \eta(Y) \subset \eta(Z)$;
(iii) $\eta \circ \eta = \eta$.

In a real linear space X, the following operators are closure operators:

$$\begin{aligned} \eta &: \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := \text{linear hull of } Y; \\ \eta &: \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := \text{affine hull of } Y; \\ \eta &: \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := coY := \text{convex hull of } Y; \end{aligned}$$

In a topological space X, the operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\eta(Y) := \overline{Y}$ is a closure operator. In a linear topological space X, the operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\eta(Y) := \overline{co}Y := \overline{coY}$ is a closure operator.

The main property of a closure operator is given by:

Lemma 3.1. *Let X be a nonempty set and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator. Let $(Y_i)_{i\in I}$ be a family of subsets of X such that $\eta(Y_i) = Y_i$ for all $i \in I$. Then*

$$\eta\left(\bigcap_{i\in I} Y_i\right) = \bigcap_{i\in I} Y_i.$$

In our considerations, in this paper, we need the following example of closure operator.

Let X_i, $i = \overline{1,m}$, be some nonempty sets and $X := \prod_{i=1}^{m} X_i$ their cartesian product. Let us denote by π_i, $i = \overline{1,m}$, the canonical projection on X_i, i.e.,

$$\pi_i : X \to X_i, \quad (x_1, \ldots, x_m) \mapsto x_i, \; i = \overline{1,m}.$$

Definition 3.2. *Let $Y \subset X$ be a subset of X. By the cartesian hull of Y we understand the subset*

$$caY := \pi_1 (Y) \times \ldots \times \pi_m (Y) .$$

Remark 3.3. In the paper [11] the set caY is denoted by $[Y]$.

Lemma 3.4. *The operator*

$$ca : \mathcal{P}(X) \to \mathcal{P}(X) , \; Y \mapsto caY$$

is a closure operator.

Proof. We remark that:
1) $Y \subset caY$, for all $Y \in \mathcal{P}(X)$;
2) $Y, Z \in \mathcal{P}(X)$, $Y \subset Z$ then $caY \subset caZ$;
3) $ca\,(caY) = caY$, for all $Y \in \mathcal{P}(X)$.

So, $ca : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator. □

Remark 3.5. $caY = Y$ if and only if Y is a cartesian product, i.e., there exists $Y_i \subset X_i$, $i = \overline{1,m}$, such that $Y = \prod_{i=1}^{m} Y_i$.

We denote by $P_{ca}(X) := \{Y \in P(X) |\; Y \text{ is cartesian set } \}$.

Remark 3.6. From Lemma 3.1 and 3.4 it follows that the intersection of an arbitrary family of cartesian sets is a cartesian set.

Lemma 3.7. *Let $Y \subset X$ be a nonempty cartesian product subset of X and $f : Y \to Y$ an operator. Then $f\,(caf\,(Y)) \subset caf\,(Y)$.*

Proof. We remark that $f\,(Y) \subset ca\,f\,(Y) \subset Y$. □

The above lemmas will be basic for our proofs.

4. Measures of noncompactness. Examples

Let (X, d) be a complete metric space and $\delta : P_b\,(X) \to \mathbb{R}_+$

$$\delta(Y) := \sup\{d(a,b)|\; a,b \in Y\}.$$

be the diameter functional on X. The Kuratowski measure of noncompactness on X is defined by $\alpha_K : P_b\,(X) \to \mathbb{R}_+$

$$\alpha_K\,(Y) := \inf \left\{ \varepsilon > 0 |\; Y = \bigcup_{i=1}^{m} Y_i,\; \delta(Y_i) \le \varepsilon,\; m \in \mathbb{N}^* \right\} .$$

The Hausdorff measure of noncompactness on X is defined by $\alpha_H : P_b(X) \to \mathbb{R}_+$

$$\alpha_H(Y) := \inf\{\varepsilon > 0 |\ Y \text{ can be covered by a finitely many balls of radius} \leq \varepsilon\}.$$

If we denote by α one of the functionals α_K and α_H then we have (see [1], [3], [5], [8], [19], [22], [4], ...):

Theorem 4.1. *The functional α has the following properties:*

(i) $\alpha(A) = 0 \Longrightarrow \overline{A}$ *is compact;*
(ii) $\alpha(A) = \alpha(\overline{A})$, $\forall A \in P_b(X)$;
(iii) $A \subset B,\ A, B \in P_b(X) \Longrightarrow \alpha(A) \leq \alpha(B)$;
(iv) *If* $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$ *and* $\alpha(A_n) \to 0$ *as* $n \to +\infty$ *then* $A_\infty := \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ *and* $\alpha(A_\infty) = 0$.

In the case of a Banach space we have that

(v) $\alpha(coA) = \alpha(A)$, $\forall A \in P_b(X)$.

Let (X, d) be a complete metric space. By definition (see [19]), a functional

$$\alpha : P_b(X) \to \mathbb{R}_+$$

is called an abstract measure of noncompactness on X iff:

(i) $\alpha(A) = 0 \Longrightarrow \bar{A}$ is compact;
(ii) $\alpha(A) = \alpha(\bar{A})$, for all $A \in P_b(X)$;
(iii) $A \subset B,\ A, B \in P_b(X) \Longrightarrow \alpha(A) \leq \alpha(B)$;
(iv) If $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$ and $\alpha(A_n) \to 0$ as $n \to +\infty$ then $A_\infty := \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ and $\alpha(A_\infty) = 0$.

In the case of a Banach space we add to these axioms the following:

(v) $\alpha(coA) = \alpha(A)$, for all $A \in P_b(X)$.

We remark that the Kuratowski's measure of noncompactness, α_K, the Hausdorff's measure of noncompactness, α_H and the diameter functional, δ, are examples of measure of noncompactness in the sense of the above definition (see [3], [7], [8], [9], [12], [19], ...). For other notions of abstract measures of noncompactness see [5], [14], [19] ...

5. Vectorial measures of noncompactness on a cartesian product of some metric spaces

Let (X_i, d_i), $i = \overline{1, m}$, be some complete metric spaces and let $X := \prod_{i=1}^{m} X_i$ their cartesian product. We consider on X the cartesian product topology. By definition a subset Y of X is a bounded subset if $\pi_i(Y) \in P_b(X_i)$, $i = \overline{1, m}$. Let α^i be a measure of noncompactness on X_i, $i = \overline{1, m}$. We consider on $P_b(X)$ the following vectorial functional

$$\alpha : P_b(X) \to \mathbb{R}^m_+,\ \alpha(Y) := \left(\alpha^1(\pi_1(Y)), \ldots, \alpha^m(\pi_m(Y))\right)^T.$$

We have:

Lemma 5.1. *The functional α has the following properties:*

(i) $Y \in P_b(X)$, $\alpha(Y) = 0 \Longrightarrow \overline{caY}$ *is compact;*
(i') $\alpha(caY) = \alpha(Y)$, *for all* $Y \in P_b(X)$*;*
(ii) $\alpha(\bar{Y}) = \alpha(Y)$, *for all* $Y \in P_b(X)$*;*
(iii) $Y \subset Z$, $Y, Z \in P_b(X) \Longrightarrow \alpha(Y) \leq \alpha(Z)$*;*
(iv) $Y_n \in P_{b,cl,ca}(X)$, $Y_{n+1} \subset Y_n$, $\alpha(Y_n) \to 0$ *as* $n \to +\infty$ *then* $Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$, $Y_\infty \in P_{b,cl,ca}(X)$ *and* $\alpha(Y_\infty) = 0$.

If $(X_i, |\cdot|_i)$, $i = \overline{1,m}$, *are Banach spaces then we have*

(v) $\alpha(coY) = \alpha(Y)$, *for all* $Y \in P_b(X)$.

Proof. The proof follows from the definition of α and from the definition of α^i. □

If we take $\alpha^i := \alpha^i_K$, $i = \overline{1,m}$, we have, by definition, the Kuratowski vectorial measure of noncompactness and if we take $\alpha^i := \alpha^i_H$, $i = \overline{1,m}$, we have the Hausdorff vectorial measure of noncompactness.

6. Fixed point theorems in terms of vectorial measures of noncompactness

Definition 6.1. *Let $S \in \mathbb{R}^{m \times m}_+$ be a matrix convergent to zero and (X_i, d_i), $i = \overline{1,m}$, complete metric spaces. Let α^i be a measure of noncompactness on X_i, $i = \overline{1,m}$, and α the corresponding vectorial measure of noncompactness on $X := \prod_{i=1}^{m} X_i$. An operator $f : X \to X$ is by definition an (α, S)-contraction iff:*

(i) $A \in P_b(X) \Longrightarrow f(A) \in P_b(X)$*;*
(ii) $\alpha(f(A)) \leq S\alpha(A)$, *for all* $A \in P_{b,ca}(X)$ *such that* $f(A) \subset A$.

If the condition (ii) is satisfied for all $A \in P_{b,ca}(X)$ then f is called a strict (α, S)-contraction.

Lemma 6.2. *Let $Y \in P_{b,cl,ca}(X)$. Let $f : Y \to Y$ be an operator such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction.*

Then, there exists $A^* \in P_{b,cl,ca}(Y)$ such that $f(A^*) \subset A^*$ and $\alpha(A^*) = 0$.

Proof. Let $Y_1 := \overline{caf(Y)}$, $Y_2 := \overline{caf(Y_1)}$, ..., $Y_{n+1} := \overline{caf(Y_n)}$, ... It is clear that $Y_n \in P_{b,cl,ca}(Y)$, $Y_{n+1} \subset Y_n$ and $f(Y_n) \subset Y_n$. Moreover, from Lemma 5.1 and (ii) we have

$$\alpha(Y_n) = \alpha\left(\overline{caf(Y_{n-1})}\right) = \alpha(f(Y_{n-1})) \leq S\alpha(Y_{n-1}) \leq \ldots \leq S^n\alpha(Y),$$

therefore, $\alpha(Y_n) \to 0$ as $n \to +\infty$. From these we have that

$$Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in P_{b,cl,ca}(Y),\ f(Y_\infty) \subset Y_\infty \text{ and } \alpha(Y_\infty) = 0.$$

So, $A^* := Y_\infty$. □

In the case of Banach spaces, if $Y \in P_{b,cl,ca,co}(Y)$ then we have in addition that $coY_\infty = Y_\infty$. In the construction of the sequence set $(Y_n)_{n\in\mathbb{N}^*}$ we take $Y_{n+1} := \overline{co\,(caf\,(Y_n))}$.

From Lemma 6.2 we have the following basic fixed point principle in the case of metric spaces:

Theorem 6.3. *Let (X_i, d_i), $i = \overline{1,m}$, be some complete metric spaces and let $X := \prod_{i=1}^{m} X_i$ their cartesian product. Let $Y \in P_{b,cl,ca}(X)$ and $f : Y \to Y$ such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction;*
(iii) *$A \in P_{b,cl,ca}(Y)$, $\alpha(A) = 0$ and $f(A) \subset A$ implies that $F_f \cap A \neq \emptyset$.*

Then

(a) *$F_f \neq \emptyset$;*
(b) *$\alpha(F_f) = 0$.*

Proof. (a) From Lemma 6.2, there exists $A^* \in P_{b,cl,ca}(Y)$ such that $f(A^*) \subset A^*$ and $\alpha(A^*) = 0$ and from condition (iii) it follows that $F_f \cap A^* \neq \emptyset$, i.e., $F_f \neq \emptyset$.

(b) We remark that $F_f \subset A^* = Y_\infty$ (see the proof of Lemma 6.2) and

$$0 \le \alpha(F_f) \le \alpha(Y_\infty) = 0.$$

□

If we take $\alpha := \delta$, the vectorial diameter functional, then from Theorem 6.3 we have:

Theorem 6.4. *Let (X_i, d_i), $i = \overline{1,m}$, be some complete metric spaces and $X := \prod_{i=1}^{m} X_i$. Let $Y \in P_{b,cl,ca}(X)$ and $f : Y \to Y$ such that:*

(i) *f is continuous;*
(ii) *f is an (δ, S)-contraction.*

Then $F_f = \{x^*\}$.

Proof. From Lemma 6.2, there exists $A^* \in P_{b,cl,ca}(Y)$ such that $f(A^*) \subset A^*$ and $\delta(A^*) = 0$. From $\delta(A^*) = 0$ we have that $A^* = \{x^*\}$ and $f(A^*) \subset A^*$ implies that $x^* \in F_f$. Also, from Theorem 6.3 we have that $\delta(F_f) = 0$, so $F_f = \{x^*\}$. □

In the case of Banach spaces we have:

Theorem 6.5. *Let $(X_i, |\cdot|_i)$, $i = \overline{1,m}$, be Banach spaces, $X := \prod_{i=1}^{m} X_i$ and $Y \in P_{b,cl,cv,ca}(X)$. Let $f : Y \to Y$ be such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction.*

Then

(a) *$F_f \neq \emptyset$;*
(b) *$\alpha(F_f) = 0$.*

Proof. Let $Y_1 := \overline{co\,(caf\,(Y))}$, $Y_2 := \overline{co\,(caf\,(Y_1))}$, ..., $Y_{n+1} := \overline{co\,(caf\,(Y_n))}$, $n \in \mathbb{N}^*$. We remark that $Y_n \in P_{b,cl,cv,ca}(Y)$, $f(Y_n) \subset Y_n$, $Y_{n+1} \subset Y_n$ and

$$\alpha(Y_n) = \alpha\left(\overline{co\,(caf\,(Y_{n-1}))}\right) = \alpha(f(Y_{n-1})) \leq S\alpha(Y_{n-1}) \leq \ldots \leq S^n\alpha(Y),$$

therefore, $\alpha(Y_n) \to 0$ as $n \to +\infty$. These imply that

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in P_{b,cl,cv,ca}(Y),\ f(Y_\infty) \subset Y_\infty \text{ and } \alpha(Y_\infty) = 0.$$

Since Y_∞ is a compact convex subset in the Banach space $X = \prod_{i=1}^{m} X_i$ (we take, for example, on X the norm $|x|_\infty = \max\{|x_1|, \ldots |x_m|\}$, which generates the cartesian product topology on X), from Schauder's fixed point theorem we have that $F_f \neq \emptyset$. But $F_f \subset Y_\infty$ is a closed subset of the compact subset Y_∞, so, F_f is a nonempty compact subset. □

For the operator $f : \prod_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i$, in the terms of vectorial norm, we have:

Theorem 6.6. *Let $(X_i, |\cdot|_i)$, $i = \overline{1,m}$, be Banach spaces, $X := \prod_{i=1}^{m} X_i$, $\|x\| := (|x_1|_1, \ldots, |x_m|_m)^T$, and $f : X \to X$ such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction;*
(iii) *there exists $T \in \mathbb{R}_+^{m\times m}$ and a vector $M \in \mathbb{R}_+^m$ such that:*
  (1) *T is a matrix convergent to zero;*
  (2) *$\|f(x)\| \leq T\|x\| + M$, for all $x \in X$.*

Then

(a) *$F_f \neq \emptyset$;*
(b) *$\alpha(F_f) = 0$.*

Proof. Let $R = (R_1, \ldots, R_m)^T \in \mathbb{R}_+^m$, with $R_i > 0$, $i = \overline{1,m}$. We denote by

$$D_R := \{x \in X \mid \|x\| \leq R\}.$$

It is clear that $D_R \in P_{b,cl,ca,co}(X)$.

First we shall prove that there exists $R^0 \in \mathbb{R}_+^m$ such that

$$f(D_R) \subset D_R,\ \forall R \in \mathbb{R}_+^m,\ R \geq R^0.$$

Let $R \in \mathbb{R}_+^m$ and $x \in D_R$, from $(iii)_{(2)}$ we have

$$\|f(x)\| \leq TR + M.$$

To prove that $f(D_R) \subset D_R$ it is sufficient to have an R such that

$$TR + M \leq R \Leftrightarrow M \leq (I_m - T)R \Leftrightarrow (I_m - T)^{-1} M \leq R.$$

So, we can take $R^0 := (I_m - T)^{-1} M$. We remark that

$$f|_{D_R} : D_R \to D_R,\ \forall R \geq R^0,$$

satisfies the conditions from the Theorem 6.5 with $Y = D_R$. □

Remark 6.7. The above results generalize some results given in [7], [16], [17], [21], [24].

Remark 6.8. For the vector-valued norm versus scalar norms see [16], [17], [20].

Remark 6.9. For the condition *(iii)* in the scalar case see [3], [8], [9], [10], [12], [13].

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