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A note on Zamfirescu's operators in Kasahara spaces

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Abstract. The aim of this paper is to give local and global fixed point results for Zamfirescu's operators in Kasahara spaces. Since the domain invariance for Zamfirescu's operators is not always satisfied, we use in our proofs the successive approximations method. Our local results extend and generalize Krasnoselskii's local fixed point theorem by replacing the context of metric space with a more general one: the Kasahara space. On the other hand, instead of contractions we use Zamfirescu's operators. As application, a homotopy result on large Kasahara spaces is given.

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1. Introduction and preliminaries

In 1972, T. Zamfirescu gives in [10] several fixed point theorems for single-valued mappings of contractive type in metric spaces, obtaining generalizations for Banach-Caccioppoli's contraction principle, Kannan's, Edelstein's and Singh's theorems. Four years later, S. Kasahara gives in [4] some generalizations of the Banach-Caccioppoli's contraction principle showing that this principle holds even if the functional d of the metric space (X, d) does not necessarily satisfy all of the axioms of the metric. In this sense S. Kasahara replaces the context of metric spaces and proves his theorems in d-complete L-spaces. In 2010, I.A. Rus introduces in [8] the notion of Kasahara space and gives similar fixed point results to those given by S. Kasahara.

In order to give the notion of Kasahara space, we recall first the notion of L-space which was given by M. Fréchet in [2].

Definition 1.1. *Let X be a nonempty set. Let*

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X)$ be a subset of $s(x)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) *if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.*
(ii) *if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and*

$$Lim(x_{n_i})_{i\in\mathbb{N}} = x.$$

Remark 1.2. For examples and more considerations on L-spaces, see [9].

We recall now the notion of Kasahara space, introduced by I.A. Rus in [8].

Definition 1.3. *Let $(X, \to)$ be an L-space and $d : X \times X \to \mathbb{R}_+$ be a functional. The triple $(X, \to, d)$ is a Kasahara space if and only if the following compatibility condition between $\to$ and d holds: for all $(x_n)_{n\in\mathbb{N}} \subset X$ with*

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty \;\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Some examples of Kasahara spaces are presented bellow.

Example 1.4 (The trivial Kasahara space). Let (X, d) be a complete metric space. Let $\overset{d}{\to}$ be the convergence structure induced by the metric d on X. Then $(X, \overset{d}{\to}, d)$ is a Kasahara space.

Example 1.5 (I.A. Rus [8]). Let (X, ρ) be a complete semimetric space (see [5], [7]) with $\rho : X \times X \to \mathbb{R}_+$ continuous. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \le cd(x, y),$ for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.

Example 1.6 (I.A. Rus [8]). Let (X, ρ) be a complete quasimetric space (see [7]) with $\rho : X \times X \to \mathbb{R}_+$. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \le cd(x, y),$ for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.

Example 1.7 (S. Kasahara [4]). Let X denote the closed interval $[0, 1]$ and $\to$ be the usual convergence structure on $\mathbb{R}$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 1, & \text{otherwise .} \end{cases}$$

Then $(X, \to, d)$ is a Kasahara space.

We recall also a very useful tool which helps us to prove the uniqueness of the fixed point for operators defined on Kasahara spaces.

Lemma 1.8 (Kasahara's lemma [4]). *Let $(X, \to, d)$ be a Kasahara space. Then*

$$d(x, y) = d(y, x) = 0 \;\Rightarrow\; x = y, \text{ for all } x, y \in X.$$

Remark 1.9. For more considerations on Kasahara spaces, see [4], [8] and the references therein.

In [10], T. Zamfirescu gives several fixed point theorems in metric spaces (X, d) for a specific contractive type operator $f : X \to X$ which satisfies at least one of the following conditions:

(i) there exists $\alpha \in [0, 1[$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X;$$

(ii) there exists $\beta \in [0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \leq \beta[d(x, f(x)) + d(y, f(y))], \text{ for all } x, y \in X;$$

(iii) there exists $\gamma \in [0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \leq \gamma[d(x, f(y)) + d(y, f(x))], \text{ for all } x, y \in X.$$

Remark 1.10. If $f : X \to X$ is a Zamfirescu operator defined on a metric space (X, d) then there exists a number $\delta \in [0, 1[$, $\delta := \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ such that at least one of the above conditions (i), (ii) and (iii) holds, where α, β and γ are replaced with δ.

In our results, the considered Zamfirescu operators are defined as follows.

Definition 1.11. *Let $(X, \to, d)$ be a Kasahara space. The mapping $f : Y \subseteq X \to X$ is called Zamfirescu operator if there exists $\delta \in [0, \frac{1}{2}[$ such that for each $x, y \in Y$ at least one of the following conditions is true:*

(1_z) $d(f(x), f(y)) \leq \delta d(x, y)$;
(2_z) $d(f(x), f(y)) \leq \delta[d(x, f(x)) + d(y, f(y))]$;
(3_z) $d(f(x), f(y)) \leq \delta[d(x, f(y)) + d(y, f(x))]$.

Throughout this paper we give some fixed point results for Zamfirescu's operators in the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a premetric. We recall the notion of premetric in the following definition.

Definition 1.12. *Let X be a nonempty set. A distance functional $d : X \times X \to \mathbb{R}_+$ is called premetric if and only if the following conditions hold:*

(1) $d(x, x) = 0$, *for all* $x \in X$;
(2) $d(x, z) \leq d(x, y) + d(y, z)$, *for all* $x, y, z \in X$.

Lemma 1.13. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. If $f : Y \subseteq X \to X$ is a Zamfirescu operator, then f has a unique fixed point.*

Proof. Let $x^*, y^* \in Y$ be two fixed points for the Zamfirescu operator f. Then $x^* = f(x^*)$ and $y^* = f(y^*)$.

Suppose that f satisfies the condition (1_z). Then we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \delta d(x^*, y^*) \Rightarrow d(x^*, y^*) = 0.$$

Similarly, we get $d(y^*, x^*) = 0$. By Kasahara's lemma 1.8, it follows that $x^* = y^*$.

Assume that f satisfies the condition (2_z). We get that

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \delta[d(x^*, f(x^*)) + d(y^*, f(y^*))] \Rightarrow d(x^*, y^*) = 0.$$

Similarly, we get $d(y^*, x^*) = 0$ and by Kasahara's lemma 1.8, it follows that $x^* = y^*$.

Finally, if f satisfies the condition (3_z), we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \delta[d(x^*, y^*) + d(y^*, x^*)].$$

Similarly, we have $d(y^*, x^*) \leq \delta[d(y^*, x^*) + d(x^*, y^*)]$.

Hence, we obtain $d(x^*, y^*) + d(y^*, x^*) \leq 2\delta[d(x^*, y^*) + d(y^*, x^*)]$ and we have further that $(1-2\delta)[d(x^*, y^*) + d(y^*, x^*)] \leq 0$. It follows that $d(x^*, y^*) = d(y^*, x^*) = 0$ and by Kasahara's lemma 1.8, we get $x^* = y^*$. □

Let $(X, \to)$ be an L-space and $f : Y \subseteq X \to X$ be an operator. The following notations and notions will be needed in the sequel of this paper:

- $F_f := \{x \in Y \mid x = f(x)\}$ the set of all fixed points for f.
- $Graph(f) := \{(x, y) \in Y \times X \mid y = f(x)\}$ the graph of f. We say that f has closed graph with respect to $\to$ or $Graph(f)$ is closed in $Y \times X$ with respect to $\to$ if and only if for any sequences $(x_n)_{n\in\mathbb{N}} \subset Y$, $(y_n)_{n\in\mathbb{N}} \subset X$ with $y_n = f(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, $y_n \to y \in X$, as $n \to \infty$, we have that $y = f(x)$.
- $\tilde{B}(x_0, r) := \big\{x \in X \mid d(x_0, x) \leq r\big\}$ the right closed ball centered in $x_0 \in X$ with radius $r \in \mathbb{R}_+$.
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for f starting from a given point $x_0 \in Y$ if $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Notice that $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$.

The aim of this paper is to give local and global fixed point results for Zamfirescu's operators in Kasahara spaces. Since the domain invariance for Zamfirescu's operators is not always satisfied, we use in our proofs the successive approximations method. Our local results extend and generalize Krasnoselskii's local fixed point theorem by replacing the context of metric space with a more general one: the Kasahara space. On the other hand, instead of contractions we use Zamfirescu's operators. As application, a homotopy result on large Kasahara spaces is given.

2. Fixed point results in Kasahara spaces

We begin this section by presenting our main local fixed point result which extends and generalizes Krasnoselskii's theorem.

Theorem 2.1 (Krasnoselskii (see e.g. [3])). *Let (X, d) be a complete metric space. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0, r) \to X$ be an operator.*

If there exists $\alpha \in [0, 1[$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$, for all $x, y \in \tilde{B}(x_0, r)$ and $d(x_0, f(x_0)) \leq (1-\alpha)r$ then f has a unique fixed point in $\tilde{B}(x_0, r)$.

Remark 2.2. Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$ and $r \in \mathbb{R}_+$. If d is continuous on X with respect to the second argument, then

(i) the right closed ball $\tilde{B}(x_0, r)$ is a closed set in X with respect to $\rightarrow$, i.e., for any sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0, r)$, with $z_n \rightarrow z \in X$, as $n \rightarrow \infty$, we get that $z \in \tilde{B}(x_0, r)$;

(ii) $(\tilde{B}(x_0, r), \rightarrow, d)$ is a Kasahara space.

Our first main result is the following.

Theorem 2.3. *Let $(X, \rightarrow, d)$ be a Kasahara space, where $d : X \times X \rightarrow \mathbb{R}_+$ is a premetric. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0, r) \rightarrow X$ be a Zamfirescu operator. We assume that:*

(i) $Graph(f)$ is closed in $\tilde{B}(x_0, r) \times X$ with respect to $\rightarrow$;

(ii) $d(x_0, f(x_0)) \leq (1 - \delta)r$;

(iii) d is continuous with respect to the second argument.

Then

(1°) f has a unique fixed point $x^* \in \tilde{B}(x_0, r)$ and $x_n := f^n(x_0) \rightarrow x^*$, as $n \rightarrow \infty$.

(2°) at least one of the following estimations holds:

$$d(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{2.1}$$

$$d(x_n, x^*) \leq \frac{\delta^n r}{(1 - 2\delta)(1 - \delta)^{n-2}}, \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

Proof. (1°). Let us consider the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$, starting from $x_0 \in X$. By the assumption (ii) it follows that the Zamfirescu operator f is a graphic contraction on $\tilde{B}(x_0, r)$.

Indeed, if f satisfies (1_z) in Definition 1.11, then by choosing $y = f(x_0) \in \tilde{B}(x_0, r)$ we have $d(f(x_0), f^2(x_0)) \leq \delta d(x_0, f(x_0))$.

If condition (2_z) is satisfied, then $d(f(x_0), f^2(x_0)) \leq \delta[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))]$ which implies that $d(f(x_0), f^2(x_0)) \leq \frac{\delta}{1-\delta} d(x_0, f(x_0))$.

If condition (3_z) is satisfied, then $d(f(x_0), f^2(x_0)) \leq \delta[d(x_0, f^2(x_0)) + d(f(x_0), f(x_0))] \leq \delta[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))]$. So we obtain again that $d(f(x_0), f^2(x_0)) \leq \frac{\delta}{1-\delta} d(x_0, f(x_0))$.

On the other hand, $d(x_0, f^2(x_0)) \leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))$ which implies further that $d(x_0, f^2(x_0)) \leq (1 - \delta^2)r$ or $d(x_0, f^2(x_0)) \leq r$, i.e., $f^2(x_0) \in \tilde{B}(x_0, r)$.

By mathematical induction, we get that for all $n \in \mathbb{N}$, $f^n(x_0) \in \tilde{B}(x_0, r)$ and that at least one of the following chains of estimations holds:

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \delta d(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \delta^n d(x_0, f(x_0)), \text{ or}$$

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \tfrac{\delta}{1-\delta} d(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \left(\tfrac{\delta}{1-\delta}\right)^n d(x_0, f(x_0)).$$

Knowing that $\delta \in [0, \frac{1}{2}[$, the series $\sum_{n\in\mathbb{N}} \delta^n$ and $\sum_{n\in\mathbb{N}} \left(\frac{\delta}{1-\delta}\right)^n$ are convergent.

It follows that $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) < +\infty$.

Since $(X, \to, d)$ is a Kasahara space, by (iii) we get that $(\tilde{B}(x_0, r), \to, d)$ is also a Kasahara space. Hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $\tilde{B}(x_0, r)$, so there exists an element $x^* \in \tilde{B}(x_0, r)$ such that $x_n \to x^*$, as $n \to \infty$.

Knowing that $Graph(f)$ is closed in $\tilde{B}(x_0, r) \times X$ with respect to $\to$, we get that $x^* \in F_f$. The uniqueness of the fixed point is assured by Lemma 1.13.

(2°). Let $p \in \mathbb{N}$, $p \geq 1$. Since

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \sum_{k=n}^{n+p-1} d(f^k(x_0), f^{k+1}(x_0))$$

we have at least one of the following two estimations:

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \delta^n \left(\sum_{k=0}^{\infty} \delta^k \right) d(x_0, f(x_0)) \leq \frac{\delta^n}{1-\delta} d(x_0, f(x_0)),$$

or $d(f^n(x_0), f^{n+p}(x_0)) \leq \left(\frac{\delta}{1-\delta}\right)^n \left[\sum_{k=0}^{\infty} \left(\frac{\delta}{1-\delta}\right)^k \right] d(x_0, f(x_0))$, that is

$d(f^n(x_0), f^{n+p}(x_0)) \leq \left(\frac{\delta}{1-\delta}\right)^n \frac{1-\delta}{1-2\delta}\, d(x_0, f(x_0))$. By letting $p \to \infty$ and by the assumption (ii), we get the estimations (2.1) and (2.2). □

We have also a global variant for Theorem 2.3.

Corollary 2.4. *Let $(X, \to, d)$ be a Kasahara space where $d : X \times X \to \mathbb{R}_+$ is a premetric, continuous with respect to the second argument. Let $f : X \to X$ be a Zamfirescu operator, having closed graph with respect to $\to$. Then*

(1°) f has a unique fixed point $x^* \in X$ and $x_n := f^n(x_0) \to x^*$, as $n \to \infty$;
(2°) for all $n \in \mathbb{N}$, at least one of the following estimations holds:

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1) \quad or \quad d(x_n, x^*) \leq \left(\frac{\delta}{1-\delta}\right)^n \frac{1-\delta}{1-2\delta}\, d(x_0, x_1).$$

Proof. Fix $x_0 \in X$ and choose $r \in \mathbb{R}_+$ such that $d(x_0, f(x_0)) \leq (1-\delta)r$. The conclusions follow from Theorem 2.3. □

Remark 2.5. Regarding the Corollary 2.4, notice that the functional d must not necessarily be a premetric in order to prove the existence of fixed points for an operator $f : X \to X$ satisfying one of the conditions (1_z) or (2_z) from the Definition 1.11. However, the functional d must be at least a premetric in the case when f satisfies condition (3_z).

Remark 2.6. The global fixed point result given in Corollary 2.4 extends and generalizes Maia's fixed point theorem (see Theorem 1 in M.G. Maia [6]) in the sense that the set X endowed with two metrics is replaced by a Kasahara space. On the other hand, Zamfirescu's operators are used instead of contractions.

The following result is a generalization of Theorem 2.3.

Corollary 2.7. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0, r) \to X$ be an operator. We consider the function $\delta : \mathbb{R}^2_+ \to [0, \frac{1}{2}[$ with $\limsup_{s \to t^+} \delta(s) < \frac{1}{2}$, for all $t \in \mathbb{R}^2_+$.*

Assume that:

(i) $Graph(f)$ is closed in $\tilde{B}(x_0, r) \times X$ with respect to $\to$;

(ii) for all $x, y \in \tilde{B}(x_0, r)$, f satisfies at least one of the following conditions:

$(1'_z)$ $d(f(x), f(y)) \leq \delta\big(d(x, y), d(y, x)\big) \cdot d(x, y)$;

$(2'_z)$ $d(f(x), f(y)) \leq \delta\big(d(x, f(x)), d(y, f(y))\big) \cdot [d(x, f(x)) + d(y, f(y))]$;

$(3'_z)$ $d(f(x), f(y)) \leq \delta\big(d(x, f(y)), d(y, f(x))\big) \cdot [d(x, f(y)) + d(y, f(x))]$;

(iii) $d(x_0, f(x_0)) \leq (1 - \delta(\cdot, \cdot))r$;

(iv) d is continuous on X with respect to the second argument.

Then the following statements hold:

(1°) f has a unique fixed point $x^* \in \tilde{B}(x_0, r)$ and $x_n := f^n(x_0) \to x^*$, as $n \to \infty$.

(2°) at least one of the relations (2.1) and (2.2) holds.

Proof. We follow the proof of Theorem 2.3. □

3. An extension to large Kasahara spaces

The aim of this section is to present an extension of our fixed point results to large Kasahara spaces. As application, a homotopy result is given. To reach our purpose, we recall first the notion of large Kasahara space.

Definition 3.1 (I.A. Rus, [8]). *Let $(X, \to)$ be an L-space, $(G, +, \leq, \xrightarrow{G})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and d_G holds:*

- *if $(x_n)_{n \in \mathbb{N}} \subset X$ is a Cauchy sequence (in a certain sense) with respect to d_G then $(x_n)_{n \in \mathbb{N}}$ converges in $(X, \to)$.*

As in the previous section, we will consider the Kasahara space $(X, \to, d)$ where $d : X \times X \to \mathbb{R}_+$ is a premetric.

In order to obtain a large Kasahara space, we need to define a certain notion of Cauchy sequence with respect to the premetric d. We must take also into account the fact that d is not symmetric.

Definition 3.2. *Let (X, d) be a premetric space with $d : X \times X \to \mathbb{R}_+$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. Then $(x_n)_{n \in \mathbb{N}}$ is a right-Cauchy sequence with respect to d if and only if*

$$\lim_{\substack{n \to \infty \\ m \to \infty}} d(x_n, x_m) = 0,$$

i.e., for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for every $m, n \in \mathbb{N}$ with $m \geq n \geq k$.

The following notion of large Kasahara space arises.

Definition 3.3. *Let $(X, \to)$ be an L-space. Let $d : X \times X \to \mathbb{R}_+$ be a premetric on X. The triple $(X, \to, d)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and d holds:*

$$\text{if } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0 \text{ then } (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Remark 3.4. Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 3.3. Then $(X, \to, d)$ is a Kasahara space.

Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty$. By following S. Kasahara (see [4]), for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, with $m > n \geq k$, we have $d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < \varepsilon$.

Hence $\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0$ and since $(X, \to, d)$ is a large Kasahara space, we get that $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. The conclusion follows from Definition 1.3.

Remark 3.5. Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 3.3. Then Theorem 2.3 and Corollaries 2.4 and 2.7 hold.

As application of Theorem 2.3 in large Kasahara spaces in the sense of Definition 3.3, we present a homotopy result which extends some similar homotopy results given on a set endowed with two metrics by A. Chiş in [1].

In our application, the following notion need to be defined.

Definition 3.6. *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 3.3. A subset U of X is an open set with respect to d if there exists a right ball $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$, $r > 0$, $x_0 \in U$ such that $B(x_0, r) \subset U$.*

Theorem 3.7. *Let $(X, \xrightarrow{\rho}, d)$ be a large Kasahara space in the sense of Definition 3.3, where $\rho : X \times X \to \mathbb{R}_+$ is a complete metric on X, $\xrightarrow{\rho}$ is the convergence structure induced by ρ on X and $d : X \times X \to \mathbb{R}_+$ is a continuous premetric on X.*

Let $Q \subset X$ be a closed set with respect to ρ. Let $U \subset X$ be an open set with respect to d and assume that $U \subset Q$.

Suppose $H : Q \times [0, 1] \to X$ satisfies the following properties:

(i) $x \neq H(x, \lambda)$ *for all* $x \in Q \setminus U$ *and all* $\lambda \in [0, 1]$*;*

(ii) for all $\lambda \in [0, 1]$ *and* $x, y \in Q$*, there exist* $\alpha \in [0, 1[$ *and* $\beta \in [0, \frac{1}{2}[$ *such that one of the following conditions holds:*

- *(ii_1)* $d(H(x, \lambda), H(y, \lambda)) \leq \alpha d(x, y)$*;*
- *(ii_2)* $d(H(x, \lambda), H(y, \lambda)) \leq \beta[d(x, H(x, \lambda)) + d(y, H(y, \lambda))]$*;*

(iii) $H(x, \lambda)$ *is continuous in* λ *with respect to* d*, uniformly for* $x \in Q$*;*

(iv) H *is uniformly continuous from* $U \times [0, 1]$ *endowed with the metric* d *on* U *into* (X, ρ)*;*

(v) *H is continuous from* $Q \times [0,1]$ *endowed with the metric* ρ *on* Q *into* (X, ρ).

In addition, assume that H_0 *has a fixed point. Then for each* $\lambda \in [0,1]$ *we have that* H_λ *has a fixed point* $x_\lambda \in U$. *(here* $H_\lambda(\cdot) = H(\cdot, \lambda)$*)*

Proof. Let $A := \{\lambda \in [0,1] \mid \text{there exists } x \in U \text{ such that } x = H(x,\lambda)\}$.

Since H_0 has a fixed point and (i) holds, we have that $0 \in A$ so the set A is nonempty. We will show that A is open and closed in $[0,1]$ and so, by the connectedness of $[0,1]$, we will have $A = [0,1]$ and the proof will be complete.

First we show that A is closed in $[0,1]$.

Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in A with $\lambda_k \to \lambda \in [0,1[$ as $k \to \infty$. By the definition of A, for each $k \in \mathbb{N}$, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$\begin{aligned} d(x_k, x_j) &= d(H(x_k,\lambda_k), H(x_j,\lambda_j)) \\ &\le d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_k,\lambda), H(x_j,\lambda)) \\ &\quad + d(H(x_j,\lambda), H(x_j,\lambda_j)) \end{aligned} \tag{3.1}$$

- If H satisfies (ii_1) then by (3.1) we get

$$\begin{aligned} &d(x_k, x_j) \le d(H(x_k,\lambda_k), H(x_k,\lambda)) + \alpha d(x_k,x_j) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\ &\Leftrightarrow (1-\alpha) d(x_k,x_j) \le d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \end{aligned}$$

- If H satisfies (ii_2) then by (3.1) we have

$$\begin{aligned} d(x_k, x_j) &\le d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\ &\quad + \beta[d(x_k, H(x_k,\lambda)) + d(x_j, H(x_j,\lambda))] \\ &= (d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\ &\quad + \beta[d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda_j), H(x_j,\lambda))]. \end{aligned}$$

By (iii), letting $k, j \to \infty$ we get that the sequence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since $(X, \overset{\rho}{\to}, d)$ is a large Kasahara space, we get that $(x_k)_{k\in\mathbb{N}}$ is convergent in $(X, \overset{\rho}{\to})$. Moreover, since $Q \subset X$ is a closed set with respect to the complete metric ρ, there exists $x \in Q$ such that $\lim_{k\to\infty} \rho(x_k, x) = 0$.

We show next that $x = H(x,\lambda)$. Indeed, we have

$$\begin{aligned} \rho(x, H(x,\lambda)) &\le \rho(x, x_k) + \rho(x_k, H(x,\lambda)) \\ &= \rho(x, x_k) + \rho(H(x_k,\lambda_k), H(x,\lambda)). \end{aligned}$$

By (v) and letting $k \to \infty$, we have $\rho(x, H(x,\lambda)) = 0$, so $x = H(x,\lambda)$ and by (i) we get that $x \in U$. Hence $\lambda \in A$ and so A is closed in $[0,1]$.

We show next that A is open in $[0,1]$.

Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. Since U is open with respect to d, by Definition 3.6 there exists a right ball $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$, $r > 0$ such that $B(x_0, r) \subset U$. By (iii), H is uniformly continuous on $B(x_0, r)$.

Let $\varepsilon = \left(1 - \max\left\{\alpha, \frac{\beta}{1-\beta}\right\}\right)r > 0$. By the uniform continuity of H, there exists $\eta = \eta(r) > 0$ such that for each $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$ we have $d(H(x,\lambda_0), H(x,\lambda)) < \varepsilon$ for any $x \in B(x_0, r)$. Since this property holds for $x = x_0$, we get $d(x_0, H(x_0,\lambda)) = d(H(x_0,\lambda_0), H(x_0,\lambda)) < \left(1 - \max\left\{\alpha, \frac{\beta}{1-\beta}\right\}\right)r$ for any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$.

By (ii), (iv) and (v) together with Theorem 2.3 in the context of large Kasahara spaces defined as in Definition 3.3, (in this case $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}\right\}$ and $f = H_\lambda$) we obtain the existence of $x_\lambda \in B(x_0, r)$ such that $x_\lambda = H_\lambda(x_\lambda)$ for any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$. Consequently A is open in $[0,1]$. □

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