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Some properties of Sobolev algebras modelled on Lorentz spaces

İlker Eryılmaz and Birsen Sağır Duyar

Abstract. In this paper, firstly Lorentz-Sobolev spaces $W^k_{L(p,q)}(\mathbb{R}^n)$ of integer order are introduced and some of their important properties are emphasized. Also, the Banach spaces $A^k_{L(p,q)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W^k_{L(p,q)}(\mathbb{R}^n)$ (Lorentz-Sobolev algebras in the sense of H.Reiter) are studied. Then, using a result due to H.C.Wang, it is showed that Banach convolution algebras $A^k_{L(p,q)}(\mathbb{R}^n)$ do not have weak factorization. Lastly, it is found that the multiplier algebra of $A^k_{L(p,q)}(\mathbb{R}^n)$ coincides with the measure algebra $M(\mathbb{R}^n)$ for $1 < p < \infty$ and $1 \le q < \infty$.

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1. Introduction

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space. If $\alpha = (\alpha_1, ..., \alpha_n)$ is an n-tuple of nonnegative integers α_j, then we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \frac{\partial}{\partial x_j}$ for $1 \le j \le n$, then

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. For given two locally integrable functions f and g on $\mathbb{R}^n$, we say that $\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = g$ (weak derivative of f) if

$$\int_{\mathbb{R}^n} f(x) \frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g(x)\, \varphi(x)\, dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the space of all smooth functions with compact support.

If we define a functional $\|\cdot\|_{k,p}$, where k is a nonnegative integer and $1 \le p \le \infty$, as follows:

$$\|f\|_{k,p} = \sum_{0\le|\alpha|\le k} \|D^\alpha f\|_p \qquad \text{if} \quad 1 \le p \le \infty, \tag{1.1}$$

for any function $f \in L^p(\mathbb{R}^n)$, then we can consider two vector spaces on which $\|\cdot\|_{k,p}$ is a norm:

(i) $W^{k,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : D^\alpha f \in L^p(\mathbb{R}^n) \text{ for } 0 \le |\alpha| \le k\}$, where $D^\alpha f$ is the weak partial derivative of f,

(ii) $W_0^{k,p}(\mathbb{R}^n) :=$ the closure of $C_0^\infty(\mathbb{R}^n)$ in the space $W^{k,p}(\mathbb{R}^n)$.

Equipped with the appropriate norm (1.1), these are called *Sobolev spaces* over $\mathbb{R}^n$. Clearly, $W^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and if $1 \le p < \infty$, $W_0^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Also, $W^{k,p}(\mathbb{R}^n)$ is a Banach space for $1 \le p \le \infty$ and a reflexive space with its associate space $W^{-k,p'}(\mathbb{R}^n)$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For any k, one can see the obvious chain of imbeddings

$$W_0^{k,p}(\mathbb{R}^n) \hookrightarrow W^{k,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n).$$

Sobolev spaces of integer order were introduced by S.L. Sobolev in [15,16]. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ by using subspaces of Lebesgue spaces. Many generalizations and specializations of these spaces have been constructed and studied in years. In particular, there are extensions that allow arbitrary real values of k, weighted spaces that introduce weight functions into the L^p-norms and other generalizations involve different orders of differentiation and different L^p-norms in different coordinate directions and Orlicz-Sobolev spaces. Finally, there has been much work on Sobolev spaces and its related areas. To an interested reader, we can suggest our main reference book [1] and the references therein.

2. Preliminaries

Definition 2.1. *Let G be a locally compact abelian group, and $(B(G), \|\cdot\|_B)$ be a Banach space of complex-valued measurable functions on G. $B(G)$ is called a homogeneous Banach space if the following are satisfied:*

H1. *$L_s f \in B(G)$ and $\|L_s f\|_B = \|f\|_B$ for all $f \in B(G)$ and $s \in G$, where $L_s f(x) = f(x-s)$.*

H2. *$s \to L_s f$ is a continuous map from G into $(B(G), \|\cdot\|_B)$.*

Definition 2.2. *A homogeneous Banach algebra on G is a subalgebra $B(G)$ of $L^1(G)$ such that $B(G)$ is itself a Banach algebra with respect to a norm $\|\cdot\|_B \ge \|\cdot\|_1$ and satisfies $H1$ and $H2$.*

Definition 2.3. *A homogeneous Banach algebra $B(G)$ is called a Segal algebra if it is dense in $L^1(G)$.*

Definition 2.4. *Let G be a locally compact abelian group with character group Γ. A Segal algebra $B(G)$ is called isometrically character-invariant if for every character $\varkappa$ and every $f \in B(G)$ one has $\varkappa f \in B(G)$ and $\|\varkappa f\|_B = \|f\|_B$. In other words, if $f \to \varkappa f$ is an isometry of $B(G)$, for all $\varkappa \in \Gamma$.*

Definition 2.5. *Let G be a locally compact abelian group with character group Γ, and μ be a positive Radon measure on Γ. A Banach algebra $(B(G), \|\cdot\|_B)$ in $L^1(G)$ is an $F^\mu-$algebra if $\widehat{B(G)} \subset L^p(G)$ for some $p \in (0,\infty)$ where "$\widehat{\ }$" denotes the Fourier transform.*

Definition 2.6. *Let G be a locally compact abelian group with character group Γ, and μ be a positive Radon measure on Γ. A Banach algebra $(B(G), \|\cdot\|_B)$ in $L^1(G)$ is a $P^\mu-$algebra if there exist two sequences (Δ_n) and (θ_n) of subsets of Γ, a sequence (f_n) in $B(G)$ and a sequence $c_n \geq 1$ satisfying*
p1. $\Delta_i \cap \Delta_j = \varnothing$ if $i \neq j$, $\theta_n \subset Int(\Delta_n)$, $\mu(\theta_n) = \alpha > 0, \mu(\Delta_n) = \beta < \infty$ for n=1,2,$\cdots$.(Int:=Interior)
p2. $0 \leq \widehat{f_n} \leq 1$, Supp$\widehat{f_n} \subset \Delta_n$, $\widehat{f_n}(\theta_n) = 1$ for each n=1,2,$\cdots$.
p3. $\|f_n\|_B \leq c_n$, $\sum_{n=1}^{\infty}\left(\frac{1}{c_n^a}\right) < \infty$, $\sum_{n=1}^{\infty}\left(\frac{1}{c_n^b}\right) = \infty$ for some $a, b \in (0,\infty)$.

An algebra is an $F^\mu P^\mu-$algebra if it is both F^μ and $P^\mu-$algebra. It is simply called $FP-$algebra if μ is the Haar measure on Γ.

Definition 2.7. *Let B be a Banach algebra. B is said to have weak factorization if, given $f \in B$, there are $f_1, \cdots, f_n, g_1, \cdots, g_n \in B$ such that $f = \sum_{i=1}^{n} f_i g_i$.*

Theorem 2.8. ([18, p.42]) *A homogeneous Banach space $(B(G), \|\cdot\|_B)$ is a homogeneous Banach algebra if and only if $B(G)$ is a linear subspace of $L^1(G)$ with $\|\cdot\|_B \geq \|\cdot\|_1$.*

Definition 2.9. *Let G be a (noncompact) locally compact abelian group. The translation coefficient K_E of a homogeneous Banach space E on G is the infimum of the constants K such that*

$$\limsup_{s\to\infty} \|f + L_s f\|_E \leq K \|f\|_E\,, \ \ \forall f \in E.$$

For the convenience of the reader, we now review briefly what we need from the theory of Lorentz spaces. Let (G, Σ, μ) be a measure space and let f be a measurable function on G. For each $y > 0$, the rearrangement of f is defined by

$$f^*(t) = \inf\{y > 0 : \mu\{x \in G : \ |f(x)| > y\} \leq t\}\,, \ t > 0,$$

where $\inf \emptyset = \infty$. Also the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds\ , \ t > 0.$$

Note that $f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous functions on $(0,\infty)$ [3, 10] .For $p, q \in (0,\infty)$, we define

$$\|f\|_{p,q}^* = \left(\frac{q}{p}\int_0^\infty [f^*(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}}, \ \ \|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty [f^{**}(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}}. \qquad (2.1)$$

Also, if $0 < p \leq \infty$ and $q = \infty$, we define

$$\|f\|_{p,\infty}^* = \sup_{t>0} t^{\frac{1}{p}} f^*(t) \ \text{ and } \ \|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t)\,.$$

For $0 < p < \infty$ and $0 < q \leq \infty$, Lorentz spaces are denoted by $L(p,q)(G)$ and defined to be the vector spaces of all measurable functions f on G such that $\|f\|_{p,q}^{*} < \infty$. We know that $\|f\|_{p,p}^{*} = \|f\|_{p}$ and so $L^{p}(G) = L(p,p)(G)$. It is also known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q}^{*} \leq \|f\|_{p,q} \leq \frac{p}{p-1}\|f\|_{p,q}^{*} \tag{2.2}$$

for each $f \in L(p,q)(G)$ and $\left(L(p,q)(G), \|\cdot\|_{p,q}\right)$ is a Banach space [3,10].

In [19], it is found that $B(p,q)(G) := L^{1}(G) \cap L(p,q)(G)$ is a normed space with the norm $\|\cdot\|_{B} = \|\cdot\|_{1} + \|\cdot\|_{p,q}$ and a Segal algebra for $1 < p < \infty$, $1 \leq q < \infty$. Nevertheless, some other properties of $B(p,q)(G)$ spaces are showed in [7].

3. The $W_{L(p,q)}^{k}(\mathbb{R}^{n})$ and $A_{L(p,q)}^{k}(\mathbb{R}^{n})$ spaces

If one looks for "*Sobolev algebras*" in literature, one sees that there are a lot of published papers about Sobolev algebras obtained by using different function spaces that are defined over different groups or sets. These spaces have been investigated under several respects, and mostly applied to the study of strongly nonlinear variational problems and partial differential equations.

In the sense of our study, we attach importance to [4-6,17]. In [5], Orlicz-Sobolev spaces that are multiplicative Banach algebras are characterized. In [6], it is showed that the space $L_{\alpha}^{p}(G) \cap L^{\infty}(G)$ is an algebra with respect to pointwise multiplication, where G is a connected unimodular Lie group. Also, sufficient conditions for the Sobolev spaces to form an algebra under pointwise multiplication have been given in [17].

In [4], Chu defined $A_{k}^{p}(\mathbb{R}^{n}) = L^{1}(\mathbb{R}^{n}) \cap W^{k,p}(\mathbb{R}^{n})$ spaces and showed some algebraic properties of these spaces (Segal algebras). In this section, we will generalize his results to Lorentz-Sobolev spaces and Lorentz-Sobolev algebras.

Definition 3.1. *Lorentz-Sobolev spaces are defined by*

$$W_{L(p,q)}^{k}(\mathbb{R}^{n}) = \{f \in L(p,q)(\mathbb{R}^{n}) : D^{\alpha}f \in L(p,q)(\mathbb{R}^{n})\} \tag{3.1}$$

for all $\alpha \in \mathbb{N}_{0}^{n}$ *with* $|\alpha| \leq k$ *where* k *is a nonnegative integer,* $p \in (1,\infty)$ *and* $q \in [1,\infty)$. *Also they are equipped with the norm*

$$\|f\|_{W_{L(p,q)}^{k}(\mathbb{R}^{n})} = \sum_{|\alpha| \leq k} \|D^{\alpha}f\|_{p,q}. \tag{3.2}$$

Clearly, if $k = 0$, then $W_{L(p,q)}^{k}(\mathbb{R}^{n}) = L(p,q)(\mathbb{R}^{n})$. Besides this, if we define $W_{L(p,q)}^{k,0}(\mathbb{R}^{n})$ as the space of the closure of $C_{0}^{\infty}(\mathbb{R}^{n})$ in the space $W_{L(p,q)}^{k}(\mathbb{R}^{n})$, then it is easy to see that $W_{L(p,q)}^{0,0}(\mathbb{R}^{n}) = L(p,q)(\mathbb{R}^{n})$ where $p \in (1,\infty)$ and $q \in [1,\infty)$. For any k, the chain of imbeddings

$$W_{L(p,q)}^{k,0}(\mathbb{R}^{n}) \hookrightarrow W_{L(p,q)}^{k}(\mathbb{R}^{n}) \hookrightarrow L(p,q)(\mathbb{R}^{n}) \tag{3.3}$$

is also clear. Instead of dealing with Lorentz-Sobolev spaces $W^k_{L(p,q)}(\mathbb{R}^n)$, we can pay attention to the completion of the set

$$\left\{f \in C^k(\mathbb{R}^n) : \|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} < \infty\right\}$$

with respect to the norm in (3.2). Because, it is easy to show that these spaces are equal.

Now, we are going to give two propositions without their (easy) proofs . One can prove them by using the same methods as those used for abstract Sobolev spaces.

Proposition 3.2. *$W^k_{L(p,q)}(\mathbb{R}^n)$ is a (homogeneous) Banach space with $\|\cdot\|_{W^k_{L(p,q)}(\mathbb{R}^n)}$.*

Proposition 3.3. *If $p, q \in (1,\infty)$, then $W^k_{L(p,q)}(\mathbb{R}^n)$ spaces are reflexive. In other words, the associate space of $W^k_{L(p,q)}(\mathbb{R}^n)$ is $W^{-k}_{L(p',q')}(\mathbb{R}^n)$ where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$.*

After this point, we are going to deal with the algebraic structures of $L^1(\mathbb{R}^n) \cap W^k_{L(p,q)}(\mathbb{R}^n)$ spaces. For this reason, we will call this intersection space as $A^k_{L(p,q)}(\mathbb{R}^n)$ and endow it with the sum norm

$$\|f\|_A := \|f\|_1 + \|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} \tag{3.4}$$

for all $f \in A^k_{L(p,q)}(\mathbb{R}^n)$.

Proposition 3.4. *$A^k_{L(p,q)}(\mathbb{R}^n)$ is a Segal algebra on $\mathbb{R}^n$ if $p \in (1,\infty)$ and $q \in [1,\infty)$.*

Proof. Let $p \in (1,\infty)$ and $q \in [1,\infty)$. Since $W^k_{L(p,q)}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ are homogeneous Banach spaces, it is easy to see that $A^k_{L(p,q)}(\mathbb{R}^n)$ is also a homogeneous Banach space under the sum norm $\|\cdot\|_A \geq \|\cdot\|_1$ by [11]. By a result of Theorem 2.8, we get $A^k_{L(p,q)}(\mathbb{R}^n)$ is a homogeneous Banach algebra. By [1, 2.19.Theorem], we know that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and is contained in $W^k_{L(p,q)}(\mathbb{R}^n)$. Therefore, $A^k_{L(p,q)}(\mathbb{R}^n)$ is a Segal algebra on $\mathbb{R}^n$. □

Theorem 3.5. *$A^k_{L(p,q)}(\mathbb{R}^n)$ is an FP-algebra for $p \in (1,\infty)$ and $q \in [1,\infty)$.*

Proof. Firstly, we are going to show the $P-$algebra property of $A^k_{L(p,q)}(\mathbb{R}^n)$ spaces.

(i) Let

$$\begin{aligned}
\Delta_m &= \left[m-\frac{1}{4}, m+\frac{1}{4}\right] \times \cdots \times \left[m-\frac{1}{4}, m+\frac{1}{4}\right] \quad (n-\text{times}) \\
\Omega_m &= \left[m-\frac{1}{8}, m+\frac{1}{8}\right] \times \cdots \times \left[m-\frac{1}{8}, m+\frac{1}{8}\right] \quad (n-\text{times})
\end{aligned}$$

and

$$\Delta'_m = \left[m-\frac{1}{4}, m+\frac{1}{4}\right],\ \Omega'_m = \left[m-\frac{1}{8}, m+\frac{1}{8}\right]$$

for $m \geq 1$. By [18, 1.8.Theorem], there exists a generalized trapezium function $f_1 \in L^1(\mathbb{R})$ such that $0 \leq \widehat{f_1} \leq 1$, supp$\widehat{f_1} \subset \Delta'_1$ and $\widehat{f_1}(\Omega'_1) = 1$.

If we let $f_m(t) = e^{i(m-1)t} f_1(t)$, then it is easy to see that $0 \leq \widehat{f_m} \leq 1$, $\mathrm{supp}\widehat{f_m} \subset \Delta'_m$ and $\widehat{f_m}(\Omega'_m) = 1$ for $m \geq 2$. If we define F_m by $F_m(x_1, ..., x_n) = f_m(x_1) \cdots f_m(x_n)$ for $m = 1, 2, ...$, then $F_m \in L^1(\mathbb{R}^n)$, $\widehat{F_m}(t_1, ..., t_n) = \widehat{f_m}(t_1) \cdots \widehat{f_m}(t_n)$ and $0 \leq \widehat{F_m} \leq 1$, $\mathrm{supp}\widehat{F_m} \subset \Delta_m$, $\widehat{F_m}(\Omega_m) = 1$. If $P(L^1(\mathbb{R}^n))$ is the set of all f in $L^1(\mathbb{R}^n)$ whose Fourier transform $\widehat{f}$ has compact support, then it is seen that $F_m \in P(L^1(\mathbb{R}^n))$. Since $P(L^1(\mathbb{R}^n))$ is dense in every homogeneous Banach algebra [18, 3.7.Theorem], we have $F_m \in A^k_{L(p,q)}(\mathbb{R}^n)$. For $1 \leq j \leq k$ and $m \geq 2$, the equality

$$\begin{aligned} f_m^{(j)}(t) &= \left(e^{i(m-1)t}\right)^{(j)} f_1(t) + \binom{j}{1}\left(e^{i(m-1)t}\right)^{(j-1)} f_1'(t) + \\ &\quad + \binom{j}{2}\left(e^{i(m-1)t}\right)^{(j-2)} f_1''(t) + ... + \binom{j}{j}\left(e^{i(m-1)t}\right) f_1^{(j)}(t) \\ &= i^j (m-1)^j e^{i(m-1)t} f_1(t) + \binom{j}{1} i^{j-1} (m-1)^{j-1} e^{i(m-1)t} f_1'(t) \\ &\quad + \binom{j}{2} i^{j-2} (m-1)^{j-2} e^{i(m-1)t} f_1''(t) + ... + \binom{j}{j}\left(e^{i(m-1)t}\right) f_1^{(j)}(t) \end{aligned}$$

is written. Since $f_m \in P(L^1(\mathbb{R})) \subset A^k_{L(p,q)}(\mathbb{R})$, if

$$M = \max\left\{\|f_1\|_{p,q},\ \|f_1'\|_{p,q}, ..., \left\|f_1^{(j)}\right\|_{p,q}\right\}$$

then, we get

$$\begin{aligned} \left\|f_m^{(j)}\right\|_{p,q} &= \left\|i^j (m-1)^j e^{i(m-1)t} f_1(t) + ... + \left(e^{i(m-1)t}\right) f_1^{(j)}(t)\right\|_{p,q} \\ &\leq (m-1)^j \|f_1\|_{p,q} + (m-1)^{j-1}\binom{j}{1}\|f_1'(t)\|_{p,q} + ... + \left\|f_1^{(j)}(t)\right\|_{p,q} \\ &\leq 2^j (m-1)^j M. \end{aligned} \tag{3.5}$$

Again, for $1 \leq |\alpha| = j \leq k$ and $0 \leq j_i \leq j$, $j_1 + \cdots + j_n = j$, it can be written by (3.5) that

$$\begin{aligned} \|D^\alpha F_m(x_1, \cdots, x_n)\|_{p,q} &= \left\|f_m^{(j_1)}(x_1) f_m^{(j_2)}(x_2) \cdots f_m^{(j_n)}(x_n)\right\|_{p,q} \\ &\leq \left(2^j (m-1)^j M\right)^n \leq \left(2^k (m-1)^k M\right)^n \end{aligned}$$

and so

$$\begin{aligned} \|F_m\|_A = \|F_m\|_1 + \|F_m\|_{W^k_{L(p,q)}(\mathbb{R}^n)} &= \|F_m\|_1 + \sum_{|\alpha| \leq k} \|D^\alpha F_m\|_{p,q} \\ = \|F_m\|_1 + \|F_m\|_{p,q} + \sum_{|\alpha|=1}\|D^\alpha F_m\|_{p,q} &+ \sum_{|\alpha|=2}\|D^\alpha F_m\|_{p,q} + \ldots + \sum_{|\alpha|=k}\|D^\alpha F_m\|_{p,q} \\ \leq \|F_m\|_1 + \|F_m\|_{p,q} &+ \left[\binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k}\right]\left(2^k (m-1)^k M\right)^n \\ &\leq B (m-1)^{kn} \end{aligned} \tag{3.6}$$

for $m \geq 2$ and some constant $B > 0$. Since we can take B and C_1 large enough such that $C_m = B(m-1)^{kn} \geq 1$ for $m = 2, 3, ...,$ $C_1 > \|F_1\|_A$ and $C_1 > 1$, we have

$$\sum\nolimits_{m=1}^{\infty} \frac{1}{C_m^{k+1}} < \infty \qquad \text{but} \qquad \sum\nolimits_{m=1}^{\infty} \frac{1}{C_m^{1/kn}} = \infty, \quad \text{for } k \geq 1.$$

Thus we get the result.

Now let $k = 0$. Then $A^0_{L(p,q)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W^0_{L(p,q)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap L(p,q)(\mathbb{R}^n) = B(p,q)(\mathbb{R}^n)$. Since $B(p,q)(\mathbb{R}^n)$ is a character invariant Segal algebra and every character Segal algebra is a $P-$algebra by [7] and [18, 4.9.Theorem], we get that $A^0_{L(p,q)}(\mathbb{R}^n)$ is a $P-$algebra.

(ii) It is obvious from (3.3) that $A^k_{L(p,q)}(\mathbb{R}^n) \subset B(p,q)(\mathbb{R}^n)$. Since $B(p,q)(\mathbb{R}^n)$ is a Segal algebra with $\widehat{B(p,q)(\mathbb{R}^n)} \subset L(p,q)(\mathbb{R}^n)$ for $p \in (1,\infty)$ and $q \in [1,\infty)$ by [3, Lemma 3.8], we get $B(p,q)(\mathbb{R}^n)$ is an $F-$algebra for $p \in (1,\infty)$ and $q \in [1,\infty)$ by [18, 4.5.Definition]. It is known from [18, Theorem 4.6] that $F-$algebra property is a going-down property. In other words, if B is an $F-$algebra and A is a subalgebra of B, then A is also an $F-$algebra. Therefore, $A^k_{L(p,q)}(\mathbb{R}^n)$ is an $F-$algebra due to $A^k_{L(p,q)}(\mathbb{R}^n) \subset B(p,q)(\mathbb{R}^n)$.

(i) and (ii) give the result. □

In [18, Theorem 8.8], it is proved that an $FP-$algebra does not admit the weak factorization property. So, we can write the following theorem.

Theorem 3.6. *$A^k_{L(p,q)}(\mathbb{R}^n)$ does not admit the weak factorization property.*

Remark 3.7. We know that a character invariant Segal algebra on the locally compact abelian group G has weak factorization if and only if it is equal to $L^1(G)$, by [8, Theorem 2.2]. For $p \in (1,\infty)$ and $q \in [1,\infty)$, we have $A^k_{L(p,q)}(\mathbb{R}^n) \subsetneqq L^1(\mathbb{R}^n)$. Therefore, an alternative proof for the preceding theorem may be done by showing character invariance of $A^k_{L(p,q)}(\mathbb{R}^n)$.

Theorem 3.8. [2] *Suppose S is a Segal algebra in $L^1(G)$ of the form $L^1(G) \cap E$, where G is a noncompact locally compact abelian group, E is a homogeneous Banach space on G. If the translation coefficient K_E of E is less than 2, then the multipliers space of S is isometrically isomorphic to the space $M(G)$ of all bounded regular Borel measures on G.*

Theorem 3.9. *The multipliers space of $A^k_{L(p,q)}(\mathbb{R}^n)$ is isometrically isomorphic to $M(\mathbb{R}^n)$ for $p \in (1,\infty)$ and $q \in [1,\infty)$.*

Proof. Let $f \in A^k_{L(p,q)}(\mathbb{R}^n)$. Then,

$$\begin{aligned} \|f + L_s f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} &= \sum_{|\alpha| \leq k} \|D^\alpha (f + L_s f)\|_{p,q} \\ &\leq \|f + L_s f\|_{p,q} + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q} + \sum_{1 \leq |\alpha| \leq k} \|L_s D^\alpha f\|_{p,q} \\ &= \|f + L_s f\|_{p,q} + 2 \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q} \end{aligned}$$

can be written. If $f=0$ (a.e.), then it is trivial that

$$\limsup_{|s|\to\infty} \|f+L_s f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} = 0.$$

Now let $f \neq 0$. Gürkanlı showed in [9, Lemma 4.1] that $K_{L(p,q)(G)} = 2^{\frac{1}{p}}$ for $p \in (1,\infty)$ and $q \in [1,\infty)$. Then, we get

$$\begin{aligned}
\limsup_{|s|\to\infty} \|f+L_s f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} &\leq \limsup_{|s|\to\infty} \|f+L_s f\|_{p,q} + 2\sum_{1\leq|\alpha|\leq k} \|D^\alpha f\|_{p,q} \\
&= 2^{\frac{1}{p}} \|f\|_{p,q} + 2\sum_{1\leq|\alpha|\leq k} \|D^\alpha f\|_{p,q} \\
&= 2^{\frac{1}{p}} \|f\|_{p,q} + 2\|f\|_{p,q} - 2\|f\|_{p,q} + 2\sum_{1\leq|\alpha|\leq k} \|D^\alpha f\|_{p,q} \\
&= \left(2^{\frac{1}{p}} - 2\right)\|f\|_{p,q} + 2\sum_{|\alpha|\leq k} \|D^\alpha f\|_{p,q} \\
&= \|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} \left(2 - \frac{\left(2-2^{\frac{1}{p}}\right)\|f\|_{p,q}}{\|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}}\right).
\end{aligned}$$

Since $0 < \|f\|_{p,q} \leq \|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}$, $0 < 2 - 2^{\frac{1}{p}} < 1$ and $0 < 2 - \frac{\left(2-2^{\frac{1}{p}}\right)\|f\|_{p,q}}{\|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}} < 2$ for all $p \in (1,\infty)$, we see that $K_{W^k_{L(p,q)}(\mathbb{R}^n)} < 2$. Therefore, the multipliers space of $A^k_{L(p,q)}(\mathbb{R}^n)$ is isometrically isomorphic to $M(\mathbb{R}^n)$ by the preceding theorem for $p \in (1,\infty)$ and $q \in [1,\infty)$. □

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İlker Eryılmaz
Ondokuz Mayıs University
Faculty of Sciences and Arts
Department of Mathematics
55139 Kurupelit-Samsun, Turkey
e-mail: `rylmz@omu.edu.tr`

Birsen Sağır Duyar
Ondokuz Mayıs University
Faculty of Sciences and Arts
Department of Mathematics
55139 Kurupelit-Samsun, Turkey
e-mail: `bduyar@omu.edu.tr`