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Approximation with an arbitrary order by generalized Szász-Mirakjan operators

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Abstract. By using two given arbitrary sequences $\alpha_n > 0$, $\beta_n > 0$, $n \in \mathbb{N}$ with the property that $\lim_{n\to\infty} \beta_n/\alpha_n = 0$, in this very short note we modify the generalized Szász-Mirakjan operator based on the Sheffer polynomials in such a way that on each compact subinterval in $[0, +\infty)$ the order of uniform approximation is $\omega_1(f; \sqrt{\beta_n/\alpha_n})$. These modified generalized operators can uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0, \infty)$ with an arbitrary good order of approximation $\sqrt{\beta_n/\alpha_n}$.

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1. Introduction

In [8], Szász introduced and investigated the approximation properties of the linear and positive operators attached to continuous functions $f : [0, \infty) \to \mathbb{R}$,

$$S_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(k/n).$$

Generalizing the above operators, in [5] Jakimovski and Leviatan introduced and studied the qualitative approximation properties of the operators given by

$$P_n(f)(x) = \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f(k/n),$$

where p_k are the Appell polynomials defined by the generating function

$$A(t)e^{tx} = \sum_{k=0}^{\infty} p_k(x)t^k, \ A(z) = \sum_{k=0}^{\infty} c_k z^k, \ c_0 \neq 0,$$

is an analytic function in a disc $|z| < R$, $(R > 1)$ and $A(1) \neq 0$. For $A(z) = 1$, one recapture the Szász-Mirakjan operators.

In [4], Ismail introduced and studied the qualitative approximation properties of a generalization of the Jakimovski-Leviatan operators, given by

$$T_n(f)(x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f(k/n),$$

where p_k are the Sheffer polynomials (more general than the Appell polynomials) defined by

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x)t^k, x \geq 0, |t| < R, \tag{1.1}$$

with $A(z) = \sum_{k=0}^{\infty} c_k z^k$, $H(z) = \sum_{k=1}^{\infty} h_k z^k$, analytic functions in a disk $|z| < R$, $(R > 1)$, $A(1) \neq 0$, $H'(1) = 1$, $c_k, h_k \in \mathbb{R}$, for all $k \geq 1$, $c_0 \in \mathbb{R}$, $c_0 \neq 0$, $h_1 \neq 0$, and supposing that $p_k(x) \geq 0$ for all $x \in [0, \infty)$, $k \geq 0$.

Quantitative estimate of the order $\omega_1(f; 1/\sqrt{n})$ in approximation by the $T_n(f)(x)$ operators were obtained by Sucu-Ibikli in [7].

By using two sequences of real numbers, $(\alpha_n)_n$, $(\beta_n)_n$ with the properties that $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$, in [1] Cetin and Ispir introduced a remarkable generalization of the Szász-Mirakjan operators attached to analytic functions f of exponential growth in a compact disk of the complex plane, $|z| < R$,

$$S_n(f; \alpha_n, \beta_n)(z) = e^{-\alpha_n z/\beta_n} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\alpha_n z}{\beta_n}\right)^k \cdot f\left(\frac{k\beta_n}{\alpha_n}\right), z \in \mathbb{C}, |z| < R,$$

which approximate f in any compact disk $|z| \leq r$, $r < R$, with the approximation order $\frac{\beta_n}{\alpha_n}$.

The main aim of this short note is to consider the Ismail's kind generalization of the above operator, but attached to a real function of real variable defined on $[0, +\infty)$,

$$T_n(f; \alpha_n, \beta_n)(x) = \frac{e^{-\alpha_n x H(1)/\beta_n}}{A(1)} \sum_{k=0}^{\infty} p_k\left(\frac{\alpha_n x}{\beta_n}\right) \cdot f\left(\frac{k\beta_n}{\alpha_n}\right), x \in [0, \infty),$$

under the above hypothesis on A, H and p_k, obtaining the order of approximation $\omega_1(f; \sqrt{\beta_n/\alpha_n})$ which, for example, in the case of Lipschitz 1 functions on $[0, \infty)$ gives the order of uniform approximation $O(\sqrt{\beta_n/\alpha_n})$ on each compact subinterval of $[0, \infty)$.

Notice that for $\alpha_n = n$, $\beta_n = 1$ for all $n \in \mathbb{N}$, we recapture the above Ismail's generalization of the Szász-Mirakjan operators. Also, evidently that $T_n(f; \alpha_n, \beta_n)(z)$ generalize the operators introduced in [1].

Since the sequence β_n/α_n, $n \in \mathbb{N}$, can evidently be chosen to converge to zero with an arbitrary small order, it seems that in the class of Szász-Mirakjan type operators, the generalization $T_n(f; \alpha_n, \beta_n)$, $n \in \mathbb{N}$, represents the best possible construction and the most general.

2. Main results

Since $T_n(f;\alpha_n,\beta_n)$, $n\in\mathbb{N}$ are positive linear operators, we will follow the standard line of study. Firstly, we need the following lemma.

Lemma 2.1. *Denoting $e_k(x)=x^k$, for all $x\in[0,\infty)$ and $n\in\mathbb{N}$ we have:*
(i) $T_n(e_0;\alpha_n,\beta_n)(x)=1$;
(ii) $T_n(e_1;\alpha_n,\beta_n)(x)=x+\frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)}{A(1)}$;
(iii) $T_n(e_2;\alpha_n,\beta_n)(x)=x^2+x\frac{\beta_n}{\alpha_n}\left(\frac{2A'(1)}{A(1)}+H''(1)+1\right)+\frac{\beta_n^2}{\alpha_n^2}\left(\frac{A''(1)+A'(1)}{A(1)}\right)$;
(iv) $T_n((\cdot-x)^2;\alpha_n,\beta_n)(x)=x\frac{\beta_n}{\alpha_n}(H''(1)+1)+\frac{\beta_n^2}{\alpha_n^2}\cdot\frac{A'(1)+A''(1)}{A(1)}$.

Proof. (i) If in (1.1) we take $t=1$ and replace x by $\frac{x\alpha_n}{\beta_n}$ then we obtain

$$A(1)\cdot e^{xH(1)\alpha_n/\beta_n}=\sum_{k=0}^{\infty}p_k(x\alpha_n/\beta_n),$$

which evidently is equivalent to $T_n(e_0;\alpha_n,\beta_n)(x)=1$.

(ii) Differentiating with respect to t the generation formula (1.1), we get

$$A'(t)\cdot e^{xH(t)}+A(t)\cdot x\cdot H'(t)\cdot e^{xH(t)}=\sum_{k=1}^{\infty}p_k(x)\cdot k\cdot t^{k-1}.$$

Taking above $t=1$ and replacing x by $x\frac{\alpha_n}{\beta_n}$, it follows

$$A'(1)\cdot e^{xH(1)\alpha_n/\beta_n}+A(1)\cdot\frac{x\alpha_n}{\beta_n}\cdot e^{xH(1)\alpha_n/\beta_n}=\sum_{k=1}^{\infty}p_k(x\alpha_n/\beta_n)\cdot k.$$

Multiplying both sides by $\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\frac{\beta_n}{\alpha_n}$, it follows

$$\frac{A'(1)}{A(1)}\cdot\frac{\beta_n}{\alpha_n}+x=\frac{e^{-x(1)\alpha_n/\beta_n}}{A(1)}\cdot\sum_{k=1}^{\infty}p_k(x\alpha_n/\beta_n)\cdot\frac{k\beta_n}{\alpha_n}=T_n(e_1;\alpha_n,\beta_n)(x).$$

(iii) Differentiating (1.1) twice with respect to t, we get

$$A''(t)e^{xH(t)}+x[2A'(t)\cdot H'(t)+A(t)H''(t)]e^{xH(t)}+x^2A(t)[H'(t)]^2\cdot e^{xH(t)}$$
$$=\sum_{k=0}^{\infty}p_k(x)k(k-1)t^{k-2}.$$

Taking here $t=1$, replacing x by $x\cdot\frac{\alpha_n}{\beta_n}$ and then multiplying both sides by $\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\frac{\beta_n^2}{\alpha_n^2}$, it follows

$$\frac{A''(1)}{A(1)}\cdot\frac{\beta_n^2}{\alpha_n^2}+x\cdot\frac{\beta_n}{\alpha_n}\left(\frac{2A'(1)}{A(1)}+H''(1)\right)+x^2$$
$$=\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\sum_{k=0}^{\infty}p_k(x\alpha_n/\beta_n)\frac{k^2\beta_n^2}{\alpha_n^2}-\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\frac{\beta n}{\alpha_n}\cdot\sum_{k=0}^{\infty}p_k(x\alpha_n/\beta_n)\frac{k\beta_n}{\alpha_n}$$
$$=T_n(e_2;\alpha_n,\beta_n)(x)-\frac{\beta_n}{\alpha_n}\cdot T_n(e_1;\alpha_n,\beta_n)(x),$$

which by using (ii) too implies

$$T_n(e_2;\alpha_n,\beta_n)(x)$$
$$= \frac{A''(1)}{A(1)}\cdot\frac{\beta_n^2}{\alpha_n^2} + x\cdot\frac{\beta_n}{\alpha_n}\left(\frac{2A'(1)}{A(1)} + H''(1)\right) + x^2 + \frac{\beta_n}{\alpha_n}\left(x + \frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)}{A(1)}\right),$$

leading to the required formula.

(iv) It is an immediate consequence of (i)-(iii) and of the linearity of $T_n(\cdot;\alpha_n,\beta_n)$. □

The main result of this section is the following.

Theorem 2.2. *Let $f : [0,\infty) \to \mathbb{R}$ be uniformly continuous on $[0,\infty)$. Denote $\omega_1(f;\delta) = \sup\{|f(x)-f(y)|; |x-y| \le \delta,\ x,y \in [0,\infty)\}$. For all $x \in [0,\infty)$, $n \in \mathbb{N}$ we have*

$$|T_n(f;\alpha_n,\beta_n)(x) - f(x)|$$
$$\le \left(1 + \sqrt{(H''(1)+1)x + \frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)+A''(1)}{A(1)}}\right)\cdot\omega_1(f;\sqrt{\beta_n/\alpha_n}).$$

Proof. By the standard theory (see e.g. Shisha-Mond [6] where although the results are obtained for continuous functions on compact intervals, the reasonings remain the same if the functions are (uniformly) continuous on $[0,+\infty)$), we have

$$|T_n(f;\alpha_n,\beta_n)(x) - f(x)| \le (1 + \delta^{-1}\sqrt{T_n((\cdot - x)^2;\alpha_n,\beta_n)(x)})\omega_1(f;\delta).$$

Replacing $\delta = \sqrt{\frac{\beta_n}{\alpha_n}}$ and using Lemma 2.1, (iv), we arrive at the desired estimate. □

As an immediate consequence of Theorem 2.2 we get the following.

Corollary 2.3. *Suppose that there exists $L > 0$ such that $|f(x)-f(y)| \le L|x-y|$, for all $x,y \in [0,\infty)$. We have*

$$|T_n(f;\alpha_n,\beta_n)(x) - f(x)|$$
$$\le L\left(1 + \sqrt{(H''(1)+1)x + \frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)+A''(1)}{A(1)}}\right)\cdot\sqrt{\beta_n/\alpha_n}.$$

Remark 2.4. In order to get uniform convergence in the above results, the expression under the square root in the above estimations must be bounded, fact which holds when x belong to a compact subinterval of $[0,+\infty)$.

Remark 2.5. The optimality of the estimates in Theorem 2.2 and Corollary 2.3 consists in the fact that given an arbitrary sequence of strictly positive numbers $(\gamma_n)_n$, with $\lim_{n\to\infty}\gamma_n = 0$, we always can find the sequences α_n, β_n satisfying $\omega_1(f;\sqrt{\beta_n/\alpha_n}) \le \gamma_n$ for all $n \in \mathbb{N}$ in the case of Theorem 2.2 and $\sqrt{\frac{\beta_n}{\alpha_n}} \le \gamma_n$ for all $n \in \mathbb{N}$, in the case of Corollary 2.3.

Remark 2.6. For $\alpha_n = n$ and $\beta_n = 1$ we recapture the results in [7], but the estimates there are essentially weaker than those in the present results.

Remark 2.7. If f is uniformly continuous on $[0, +\infty)$ then it is well known that its growth on $[0, +\infty)$ is linear, i.e. there exist $\alpha, \beta > 0$ such that $|f(x)| \leq \alpha x + \beta$, for all $x \in [0, +\infty)$ (see e.g. [2], p. 48, Problème 4, or [3]).

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