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Higher order iterates of Szasz-Mirakyan-Baskakov operators

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Abstract. In this paper, we discuss the generalization of Szasz-Mirakyan-Baskakov type operators defined in [7], using the iterative combinations in ordinary and simultaneous approximations. We have better estimates in higher order modulus of continuity for these operators in simultaneous approximation.

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1. Introduction

Lebesgue integrable functions f on $[0,\infty)$ are defined by

$$H[0,\infty)=\left\{f:\int_0^\infty \frac{|f(t)|}{(1+t)^n}dt<\infty, n\in N\right\}$$

A new sequence of linear positive operators was introduced by Gupta-Srivastava [4] in 1995. They combined Szasz-Mirakyan and Baskakov operators as

$$S_n(f;x)=(n-1)\sum_{v=0}^{\infty}q_{n,v}(x)\int_0^\infty p_{n,v}(t)f(t)dt, \quad \forall x\in[0,\infty), \tag{1.1}$$

where

$$\begin{aligned} p_{n,v}(t) &= \frac{(n+v-1)!}{v!(n-1)!}\frac{t^v}{(1+t)^{n+v}}, \\ q_{n,v}(x) &= \frac{e^{-nx}(nx)^v}{v!}, \quad 0\le x<\infty. \end{aligned}$$

We define the norm $\|.\|$ on $C_\gamma[0,\infty)$ by

$$\|f\|_\gamma=\sup_{0\le t<\infty}|f(t)|t^{-\gamma},$$

where $C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le Mt^\gamma, \gamma > 0\}$. It can be noticed that the order of approximation by these operators (1.1) is at best of $O(n^{-1})$, howsoever smooth the function may be. So in order to improve the rate of convergence, we consider the iterative combinations $R_{n,v} : H[0,\infty) \to C^\infty[0,\infty)$ of the operators $S_n(f,x)$ described as below

$$R_{n,v}(f(t),x) = (I-(I-S_n)^v)(f;x) = \sum_{r=1}^{v}(-1)^{r+1}\binom{v}{r} S_n^r(f(t);x), \qquad (1.2)$$

where $S_n^0 = I$ and $S_n^r = S_n(S_n^{r-1})$ for $r \in N$.

The purpose of this paper is to obtain the corresponding general results in terms of $(2k+2)^{th}$ order modulus of continuity by using properties of linear approximating method, namely Steklov Mean. In the present paper, we use the notations

$$\begin{aligned} I &\equiv [a,b], \quad 0 < a < b < \infty, \\ I_i &\equiv [a_i,b_i], \quad 0 < a_1 < a_2 < \ldots < b_2 < b_1 < \infty;\ i = 1,2,\ldots \end{aligned}$$

Also $\|.\|_{C(I)}$ is sup-norm on the interval I and having not same value in different cases by constant C. Some approximation properties for similar type operators were discussed in [3] and [7]. Very recently D. Sharma et al [8] obtained some results on similar type of operators.

2. Auxiliary results

In this section, we obtain some important lemmas which will be useful for the proof of our main theorem.

Lemma 2.1. [6] *For $m \in N^0$, we define*

$$U_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x)\left(\frac{v}{n} - x\right)^m,$$

then $U_{n,0} = 1$, $U_{n,1} = 0$. Further, there holds the recurrence formula

$$nU_{n,m+1}(x) = x\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right], \quad m \ge 1.$$

Consequently

1. *$U_{n,m}(x)$ is a polynomial in x of degree $\le m$.*
2. *$U_{n,m}(x) = O(n^{-[m+1]/2})$, where $[\zeta]$ is integral part of ζ.*

Lemma 2.2. [4] *There exists the polynomials $\phi_{i,j,r}(x)$ independent of n and v such that*

$$x^r(1+x)^r \frac{d^r}{dx^r} p_{n,v}(x) = \sum_{\substack{2i+j\le r;\\ i,j\ge 0}} n^i (v-nx)^j \phi_{i,j,r}(x) p_{n,v}(x);$$

$$x^r \frac{d^r}{dx^r} q_{n,v}(x) = \sum_{\substack{2i+j\le r;\\ i,j\ge 0}} n^i (v-nx)^j \phi_{i,j,r}(x) q_{n,v}(x).$$

Lemma 2.3. [3] *We assume that* $0 < a_1 < a_2 < b_2 < b_1 < \infty$, *for sufficiently small* $\delta > 0$, *then* $(2k+2)^{th}$ *ordered Steklov mean* $g_{2k+2,\delta}(t)$ *which corresponds to* $g(t) \in C_\gamma[0,\infty)$, *is defined as*

$$g_{2k+2,\delta}(t) = \delta^{-(2k+2)} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \cdots \int_{-\delta/2}^{\delta/2} [g(t) - \Delta_\eta^{2k+2} g(t)] \prod_{i=1}^{2k+2} dt_i,$$

where

$$\eta = \frac{1}{2k+2} \sum_{i=1}^{2k+2} t_i, \quad \forall t \in [a,b].$$

It is easily checked in [1], [2] *and* [5] *that*

1. $g_{2k+2,\delta}$ *has continuous derivatives upto order* $(2k+2)$ *on* $[a,b]$;
2. $\|g^{(r)}_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K\delta^{-r}\omega_r(g,\delta,a,b), \quad r = 1,2,...(2k+2)$;
3. $\|g - g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K\omega_{2k+2}(g,\delta,a,b)$;
4. $\|g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K\|g\|_\gamma$.

Here K *is a constant not necessarily same at different places.*

Lemma 2.4. *For the* m^{th} *order moment* $T_{n,m}(x), m \in N^0$ *defined by*

$$T_{n,m}(x) = (n-1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_0^\infty p_{n,v}(t)(t-x)^m dt, \qquad \forall x \in [0,\infty)$$

we obtain

$$T_{n,0}(x) = 1 \tag{2.1}$$

$$T_{n,1}(x) = \frac{1+2x}{n-2}, \qquad n > 2 \tag{2.2}$$

$$T_{n,2}(x) = \frac{(n+6)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}, \qquad n > 3 \tag{2.3}$$

and the recurrence relation for $n > (m+2)$

$$(n-m-2)T_{n,m+1}(x) = x[T'_{n,m}(x) + m(2+x)T_{n,m-1}(x)] + (m+1)(1+2x)T_{n,m}(x) \tag{2.4}$$

Further, for all $x \in [0,\infty)$, *we have* $T_{n,m}(x) = O(n^{-[m+1]/2})$.

Proof. Obviously (2.1)-(2.3) can be easily proved by using the definition of $T_{n,m}(x)$. To prove the recurrence relation (2.4), we proceed by taking

$$T'_{n,m}(x) = (n-1) \sum_{v=0}^{\infty} q'_{n,v}(x) \int_0^\infty p_{n,v}(t)(t-x)^m dt - mT_{n,m-1}(x)$$

Multiplying by x on both sides and then using identity $xq'_{n,v}(x) = (v - nx)q_{n,v}(x)$, we have

$$\begin{aligned}
x[T'_{n,m}(x) + mT_{n,m-1}(x)] &= (n-1)\sum_{v=0}^{\infty}(v-nx)q_{n,v}(x)\int_0^{\infty} p_{n,v}(t)(t-x)^m dt \\
&= (n-1)\sum_{v=0}^{\infty}(v-nx)q_{n,v}(x)\int_0^{\infty} p_{n,v}(t)(t-x)^m dt \\
&= (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} (v-nx)p_{n,v}(t)(t-x)^m dt \\
&= (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} (v-nt)p_{n,v}(t)(t-x)^m dt \\
&\quad +(n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} np_{n,v}(t)(t-x)^{m+1} dt \\
&= (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} (v-nt)p_{n,v}(t)(t-x)^m dt \\
&\quad +nT_{n,m+1}(x).
\end{aligned}$$

Again, using identity $t(1+t)p'_{n,v}(t) = (v - nt)p_{n,v}(t)$ in RHS, we get

$$\begin{aligned}
& x[T'_{n,m}(x) + mT_{n,m-1}(x)] \\
= \;& (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} t(1+t)p'_{n,v}(t)(t-x)^m dt + nT_{n,m+1}(x) \\
= \;& (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} [(1+2x)(t-x)+(t-x)^2+x(1+x)]p'_{n,v}(t) \\
& \times(t-x)^m dt + nT_{n,m+1}(x) \\
= \;& -(m+1)(1+2x)T_{n,m}(x) - (m+2)T_{n,m+1}(x) - mx(1+x)T_{n,m-1}(x) \\
& +nT_{n,m+1}(x).
\end{aligned}$$

This leads to our required result (2.4).

Further, for every $m \in N^0$, the m^{th} order moment $T^{(p)}_{n,m}$ for the operator S^p_n is defined by

$$T^{(p)}_{n,m}(x) = S^p_n((t-x)^m, x).$$

If we adopt the convention $T^{(1)}_{n,m}(x) = T_{n,m}(x)$, obviously $T^{(p)}_{n,m}(x)$ is of degree m.

Theorem 2.5. *Let $f \in C_\gamma[0,\infty)$, if $f^{(2v+p+2)}$ exists at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n^{v+1}[R^{(p)}_{n,v}(f;x) - f^{(p)}(x)] = \sum_{k=p}^{2v+p+2} Q(k,v,p,x)f^{(k)}(x), \tag{2.5}$$

where $Q(k,v,p,x)$ are certain polynomials in x. Further if $f^{(2v+p+2)}$ is continuous on $(a-\eta, b+\eta) \subset [0,\infty)$ and $\eta > 0$, then this theorem holds uniformly in $[a,b]$.

Proof will be along the similar lines [4].

3. Main results

In this section, we establish the direct theorem.

Theorem 3.1. *Let $f \in H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(t^\alpha)$ as $t \to \infty$ for some $\alpha > 0$.*
If $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset [0,\infty)$, for some $\eta > 0$. Then

$$\|R_{n,v}^{(p)}(f(t);x) - f^{(p)}(x)\|_{C(I_2)} \le K\{n^{-v}\|f\|_{C(I_1)} + \omega_{2v+2}(f^{(p)}, n^{-1/2}, I_1)\}$$

where constant K is independent of f and n.

Proof. We can write

$$\begin{aligned} &\left\|R_{n,v}^{p}(f(t);x) - f^{(p)}(x)\right\|_{C(I_2)} \\ \le\ & \|R_{n,v}^{(p)}(f - f_{\eta,2v+2};x)\|_{C(I_2)} + \left\|R_{n,v}^{(p)}(f_{\eta,2v+2};x) - f_{\eta,2v+2}^{(p)}(x)\right\|_{C(I_2)} \\ &+ \left\|f^{(p)}(x) - f_{\eta,2v+2}^{(p)}(x)\right\|_{C(I_2)} \\ =:\ & P_1 + P_2 + P_3. \end{aligned}$$

By the property of Steklov Mean and $f_{\eta,2v+2}^{(p)}(x) = (f^{(p)})_{\eta,2v+2}(x)$, we get

$$P_3 \le K\omega_{2v+2}(f^{(p)}, \eta, I_1).$$

To estimate P_2, applying Theorem 2.5 and interpolation property from [2], we have

$$\begin{aligned} P_2 &\le Kn^{-(v+1)} \sum_{i=p}^{2v+p+2} \|f_{\eta,2v+2}^{(i)}(x)\|_{C(I_2)} \\ &\le Kn^{-(v+1)} \left(\|f_{\eta,2v+2}\|_{C(I_2)} + \|(f_{\eta,2v+2}^{(p)})^{(2v+2)}\|_{C(I_2)}\right). \end{aligned}$$

Hence by using properties (2) and (4) of Steklov Mean, we get

$$P_2 \le Kn^{-(v+1)}[\|f\|_{C(I_1)} + (\eta)^{-2v-2}\omega_{2v+2}(f^{(p)}, \eta, I_1)].$$

Suppose a^* and b^* be such that

$$0 < a_1 < a^* < a_2 < b_2 < b^* < b_1 < \infty.$$

In order to estimate P_1, let $F = f - f_{\eta,2v+2}$. Then, by hypothesis, we have

$$F(t) = \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!}(t-x)^i + \frac{F^{(p)}(\xi) - F^{(p)}(x)}{p!}(t-x)^p\psi(t) + h(t,x)(1-\psi(t)), \quad (3.1)$$

where ξ lies between t and x, and ψ is the characteristic function of the interval $[a^*, b^*]$. For $t \in [a^*, b^*]$ and $x \in [a_2, b_2]$, we get

$$F(t) = \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!}(t-x)^i + \frac{F^{(p)}(\xi) - F^{(p)}(x)}{p!}(t-x)^p,$$

and for $t \in [0,\infty) \setminus [a^*, b^*]$, $x \in [a_2, b_2]$, we define

$$h(t,x) = F(t) - \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!}(t-x)^i.$$

Now operating $R_{n,v}^p$ on both the sides of (3.1), we have the three terms on right side namely E_1, E_2 and E_3 respectively. By using (1.2) and Lemma 2.4, we get

$$\begin{aligned} E_1 &= \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!} \sum_{r=1}^{v} (-1)^{r+1} \begin{pmatrix} v \\ r \end{pmatrix} D^p \left(S_n^r((t-x)^i; x\right), \quad D \equiv \frac{d}{dx} \\ &= \frac{F^{(p)}(x)}{p!} \sum_{r=1}^{v} (-1)^{r+1} \begin{pmatrix} v \\ r \end{pmatrix} D^p \left(S_n^r(t^p; x)\right) \\ &\to F^{(p)}(x), \end{aligned}$$

when $n \to \infty$ uniformly in I_2. Therefore

$$\|E_1\|_{C(I_2)} \le K \left\| f^{(p)} - f_{\eta,2v+2}^{(p)} \right\|_{C(I_2)}.$$

To obtain E_2, we have

$$\begin{aligned} \|E_2\|_{C(I_2)} &\le \frac{2}{p!} \left\| f^{(p)} - f_{\eta,2v+2}^{(p)} \right\|_{C[a^*,b^*]} \sum_{r=1}^{v} (n-1) \begin{pmatrix} v \\ r \end{pmatrix} \sum_{v=0}^{\infty} |q_{n,v}^{(p)}(x)| \\ &\quad \times \int_0^\infty p_{n,v}(t) S_n^{r-1}\left(|t-x|^p, x\right) dt. \end{aligned}$$

Using Lemma 2.2, Cauchy Schwartz Inequality and Lemma 2.1

$$\begin{aligned} &(n-1)\sum_{v=0}^{\infty} |q_{n,v}^{(p)}(x)| \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t-x|^p, x) dt \\ \le\ & K \sum_{\substack{2r+j\le p;\\ r,j\ge 0}} n^r \phi_{r,j,p}(x) x^{-p} (n-1) \sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^j \\ &\times \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t-x|^p, x) dt \\ \le\ & K \sum_{\substack{2r+j\le p;\\ r,j\ge 0}} n^r \left(\sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j} \right)^{1/2} \\ &\times \left((n-1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t-x|^{2p}, x) dt \right)^{1/2} \\ =\ & K \sum_{\substack{2r+j\le p;\\ r,j\ge 0}} n^r O(n^{j/2}) O(n^{-p/2}) \\ =\ & O(1) \end{aligned}$$

as $n \to \infty$ uniformly in I_2. Therefore,

$$\|E_2\| \leq K\|f^{(p)} - f^{(p)}_{\eta,2v+2}\|_{C(a^*,b^*)}.$$

Since $t \in [0,\infty) \setminus [a^*,b^*]$ and $x \in [a_2,b_2]$, we can choose a $\delta > 0$ in such a way that $|t-x| \geq \delta$. If $\beta \geq \max\{\alpha, p\}$ be an integer, we can find a positive constant Q such that $|h(t,x)| \leq Q|t-x|^\beta$ whenever $|t-x| \geq \delta$. Again applying Lemma 2.2, Cauchy Schwartz Inequality three times, Lemma 2.1 and Lemma 2.4, we get

$$\begin{aligned}
|E_3| &\leq K\sum_{r=0}^{v}\binom{v}{r}\sum_{\substack{2r+j\leq p;\\ r,j\geq 0}} n^r \left(\sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j}\right)^{1/2} \\
&\quad\times\left((n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^\infty p_{n,v}(t)S_n^{r-1}((1-\psi(t))(t-x)^{2\beta};t)dt\right)^{1/2} \\
&\leq K\sum_{r=0}^{v}\binom{v}{r}\sum_{\substack{2r+j\leq p;\\ r,j\geq 0}} n^r \left(\sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j}\right)^{1/2} \\
&\quad\times\left((n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^\infty p_{n,v}(t)S_n^{r-1}(\frac{(t-x)^{2m}}{\delta^{2m-2\beta}};t)dt\right)^{1/2} \\
&\leq K\sum_{\substack{2r+j\leq p;\\ r,j\geq 0}} n^r O(n^{j/2})O(n^{-m/2}), \qquad m > \beta, \forall m \in I.
\end{aligned}$$

Hence $\|E_3\| = O(1)$, as $n \to \infty$, uniformly in I_2. Combining the estimates of E_1, E_2 and E_3, we get

$$P_1 \leq K\|f^{(p)} - f^{(p)}_{\eta,2v+2}\|_{C(a*,b*)} \leq K\omega_{2v+2}(f^{(p)},\eta,I_1).$$

Substituting $\eta = n^{-1/2}$, we get the required theorem.

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