

# On the global uniform asymptotic stability of time-varying dynamical systems

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**Abstract.** The objective of this work is twofold. In the first part, we present sufficient conditions for global uniform asymptotic stability and/or practical stability in terms of Lyapunov-like functions for nonlinear time varying systems. Furthermore, an illustrative numerical example is presented.

**Mathematics Subject Classification (2010):** 34D20, 37B25, 37B55.

**Keywords:** Nonlinear time-varying systems, Lyapunov function, asymptotic stability.

## 1. Introduction

The problem of stability analysis of nonlinear time-varying systems has attracted the attention of several researchers and has produced a vast body of important results [1, 2, 13, 11] and the references therein.

This fact motivated to study systems whose desired behavior is asymptotic stability about the origin of the state space or a close approximation to this, e.g., all state trajectories are bounded and approach a sufficiently small neighborhood of the origin [5] and references therein. Quite often, one also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. To this end, the authors of [6] introduce a concept of exponential rate of convergence and for a specific class of uncertain systems they present controllers which guarantee this behavior. This property is referred to us as practical stability (see [4] for the more explanation and [3]).

On the other hand, the asymptotic stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, thus the notion of practical stability is more suitable in several situations than Lyapunov stability (see [9], [10]).

This paper aims, as first objective, to provide sufficient conditions that ensure the global uniform practical stability of system (2.1). A new quick proof for the results of [4] is also presented. Next, sufficient conditions for the GUPAS are presented.

Moreover, an example in dimensional two is given to illustrate the applicability of the result.

## 2. Definitions and tools

Consider the time varying system described by the following:

$$\dot{x} = f(t, x) + g(t, x) \quad (2.1)$$

where  $f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are piecewise continuous in  $t$  and locally Lipschitz in  $x$  on  $\mathbb{R}^+ \times \mathbb{R}^n$ . We consider also the associated nominal system

$$\dot{x} = f(t, x) \quad (2.2)$$

For all  $x_0 \in \mathbb{R}^n$  and  $t_0 \in \mathbb{R}$ , we will denote by  $x(t; t_0, x_0)$ , or simply by  $x(t)$ , the unique solution of (2.1) at time  $t_0$  starting from the point  $x_0$ .

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation,  $\|\cdot\|$  stands for the Euclidean norm vectors. We recall now some standard concepts from stability and practical stability theory; any book on Lyapunov stability can be consulted for these; particularly good references are [7, 8]:  $\mathcal{K}$  is the class of functions  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$  which are zero at the origin, strictly increasing and continuous.  $\mathcal{K}_\infty$  is the subset of  $\mathcal{K}$  functions that are unbounded.  $\mathcal{L}$  is the set of functions  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$  which are continuous, decreasing and converging to zero as their argument tends to  $+\infty$ .  $\mathcal{KL}$  is the class of functions  $\mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which are class  $\mathcal{K}$  on the first argument and class  $\mathcal{L}$  on the second argument. A positive definite function  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$  is one that is zero at the origin and positive otherwise. We define the closed ball  $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$ .

In order to simplify the notation, we use the following notation.

$$\nabla_t V + \nabla_x^\top V \cdot f(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x) \quad (2.3)$$

where  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ .

Next, we need the definitions given below.

### Definition 2.1 (uniform stability of $B_r$ ).

1. The solutions of system (2.1) are said to be uniformly bounded if for all  $\alpha$  and any  $t_0 \geq 0$  there exists a  $\beta(\alpha) > 0$  so that  $\|x(t)\| < \beta$  for all  $t \geq t_0$  whenever  $\|x_0\| < \alpha$ .
2. The ball  $B_r$  is said to be uniformly stable if for all  $\varepsilon > r$ , there exists  $\delta := \delta(\varepsilon)$  such that for all  $t_0 \geq 0$

$$\|x_0\| < \delta \implies \|x(t)\| < \varepsilon, \quad \forall t \geq t_0. \quad (2.4)$$

3.  $B_r$  is globally uniformly stable if it is uniformly stable and the solutions of system (2.1) exist and are globally uniformly bounded.

**Definition 2.2 (uniform attractivity of  $B_r$ ).**  $B_r$  is globally uniformly attractive if for all  $\varepsilon > r$  and  $c > 0$ , there exists  $T =: T(\varepsilon, c) > 0$  such that for all  $t_0 \geq 0$ ,

$$\|x(t)\| < \varepsilon \quad \forall t \geq t_0 + T, \quad \|x_0\| < c.$$

System (2.2) is globally uniformly practically asymptotically stable (GUPAS) if there exists  $r \geq 0$  such that  $B_r$  is globally uniformly stable and globally uniformly attractive.

Sufficient condition for GUPAS is the existence of a class  $\mathcal{KL}$  function  $\beta$  and a constant  $r > 0$  such that, given any initial state  $x_0$ , ensuing trajectory  $x(t)$  satisfies:

$$\|x(t)\| \leq r + \beta(\|x_0\|, t), \quad \forall t \geq t_0. \quad (2.5)$$

If the class  $\mathcal{KL}$  function  $\beta$  on the above relation (2.5) is of the form  $\beta(r, t) = kr^{-\lambda t}$ , with  $\lambda, k > 0$  we say that the system (2.2) is globally uniformly practically exponentially stable (GUPES). It is also, worth to notice that if in the above definitions, we take  $r = 0$ , then one deals with the standard concept of GUAS and GUES. Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for  $0 \leq \|x(t)\| - r$ , so that if  $r = 0$  in the above definitions we find the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point (see [8] for more details).

In the sequel and in the order to solve the problem of GUAS and uniform exponential convergence to the ball  $B_r$  of the perturbed system (2.1), we introduce two technical lemmas, where the proof of the second one is given in appendixes, that will be crucial in establishing the main result of this work.

**Lemma 2.3.** [12] *Let  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  be a continuous function,  $\varepsilon$  is a positive real number and  $\lambda$  is a strictly positive real number. Assume that for all  $t \in [0, +\infty)$  and  $0 \leq u \leq t$ , we have*

$$\varphi(t) - \varphi(u) \leq \int_u^t (-\lambda\varphi(s) + \varepsilon) ds. \quad (2.6)$$

Then

$$\varphi(t) \leq \frac{\varepsilon}{\lambda} + \varphi(0) \exp(-\lambda t). \quad (2.7)$$

**Lemma 2.4.** *Let  $y : [0, +\infty[ \rightarrow [0, +\infty[$  be a differentiable function,  $\alpha$  be a class  $\mathcal{K}_\infty$  function and  $c$  be a positive real number. Assume that for all  $t \in [0, +\infty[$  we have,*

$$\dot{y}(t) \leq -\alpha(y(t)) + c. \quad (2.8)$$

Then, there exists a class  $\mathcal{KL}$  function  $\beta_\alpha$  such that

$$y(t) \leq \alpha^{-1}(2c) + \beta_\alpha(y(0), t). \quad (2.9)$$

*Proof.* Let  $\alpha$  of class  $\mathcal{K}_\infty$  and  $c$  be such that (2.8) holds.

Since  $\alpha$  is a class  $\mathcal{K}_\infty$  function, then there exists some constant  $d$  such that  $\alpha(d) = 2c$ . Using this, one sees that

$$\dot{y}(t) \leq -\frac{\alpha(y(t))}{2}, \quad \text{whenever } y(t) \geq d. \quad (*)$$

Let us introduce the set

$$S = \{y : y(t) \leq d\}.$$

We claim that the set  $S$  is forward invariant. That is to say that if  $x(t_0) \in S$  for some  $t_0 \geq 0$ , then  $x(t) \in S$  for all  $t \geq t_0$ . Indeed, suppose to the contrary that there exists  $a > t_0$  such that  $y(a) > d$ .

Consider the set

$$\Delta = \{x < a \in \mathbb{R}_+ \text{ such that } y(t) > d, \forall t \in ]x, a[ \}$$

Since  $t_1 \in \Delta$  and the continuity of  $y$  on  $\mathbb{R}_+$ , we have  $\Delta \neq \emptyset$ . Also, since  $\Delta$  is lower bounded by  $t_0$  then,  $m = \inf \Delta$  is finite.

We know that  $m \in \bar{\Delta}$ , where  $\bar{\Delta}$  denotes the closure of  $\Delta$ . So that, we get by continuity  $y(m) \geq d$ ,  $m \geq t_0$  moreover, we have  $y(t) > d$  for  $t \in ]m, a[$ . It follows that the inequality in the left hand side of (\*) holds for each  $t \in ]m, a[$ , and therefore that the absolutely continuous function  $y(t)$  has a negative derivative almost everywhere on the interval  $]m, a[$ . Thus  $y(a) \leq y(t_0) \leq d$ . This contradicts the fact that  $y(a) > d$ . So  $S$  must indeed be forward invariant, as claimed.

We continue now the proof of the Lemma. We distinguish the two possible cases:

- If  $y(0) \leq d$ , then  $y(t) \leq d$  for all  $t \geq 0$  as claimed above.
- If  $y(0) > d$ , then there exists  $t_1 > 0$  (possibly  $t_1 = +\infty$ ) such that  $y(t) > d$ , for all  $t \in [0, t_1[$  and  $y(t) \leq d$ , for all  $t \geq t_1$ , with the understanding that the second case does not happen if  $t_1 = +\infty$ .

Now we apply Lemma 2.2 in [14], to obtain a class  $\mathcal{KL}$  function  $\beta_\alpha$  such that  $y(t) \leq \beta_\alpha(y(0), t), \forall t \in [0, t_1[$ .

So every where we have,

$$y(t) \leq d + \beta_\alpha(y(0), t) = \alpha^{-1}(2c) + \beta_\alpha(y(0), t).$$

which completes the proof.

### 3. Main results

In what follows, with the aid of the previous Lemmas we give some new results on GUAS and practical stability.

#### 3.1. Global uniform asymptotic stability

In this section we suppose that the origin  $x = 0$  is equilibrium point for system (2.2) and the perturbation  $g$  vanishes, that is  $g(t, 0) = 0, \forall t \geq 0$ .

Consider the nonlinear system (2.2) and introduce a set of assumptions.

Assume that:

- (A1). There exists continuously differentiable  $V : [0, +\infty[ \times \mathbb{R}^n \rightarrow [0, +\infty[$ , such that

$$c_1 \|x\|^c \leq V(t, x) \leq c_2 \|x\|^c \quad (3.1a)$$

$$\nabla_t V + \nabla_x^\top V f(t, x) \leq -c_3 \|x\|^c \quad (3.1b)$$

$$|\nabla_x^\top V g(t, x)| \leq a(\|x\|)b(t) \quad (3.1c)$$

where  $c_1, c_2, c_3$  and  $c$  are strictly positive constants,  $a : [0, +\infty[ \rightarrow \mathbb{R}$  and  $b : [0, +\infty[ \rightarrow \mathbb{R}$  are continuous functions satisfying:

$$\int_0^\infty b(s) ds \leq b_\infty \quad \lim_{t \rightarrow \infty} b(t) = 0 \tag{3.2}$$

for some constant  $b_\infty$ .

**(A2).** There exist some constants  $k > 0$  and  $r \geq 0$ , such that

$$a(\|x\|) \leq k\|x\|^c, \quad \text{for all } \|x\| \geq r. \tag{3.3}$$

We state the following result.

**Proposition 3.1.** *Under assumptions (A1) and (A2), the equilibrium point  $x = 0$  of (2.1) is globally uniformly asymptotically stable.*

*Proof.* We first prove the forward completeness of system (2.1) by proving that the finite escape time phenomenon does not occur, secondly we prove global uniform asymptotic stability in the sense of Lyapunov.

*No finite escape time:*

The time derivative of  $V(t, x)$  along the trajectories of system (2.1) is given by:

$$\dot{V}(t, x) = \nabla_t V + \nabla_x^\top V \cdot f(t, x) + \nabla_x^\top g(t, x) \tag{3.4}$$

From inequalities (3.1a), (3.1b) and (3.1c), we have

$$\begin{aligned} \|x(t)\|^c &\leq \frac{1}{c_1} V(0, x(0)) + \frac{1}{c_1} \int_0^t \dot{V}(s, x(s)) ds \\ &\leq \frac{c_2}{c_1} \|x(0)\|^c + \frac{1}{c_1} \int_0^t b(s) a(\|x(s)\|) ds. \end{aligned}$$

Let

$$\alpha_r = \sup_{\|x\| \leq r} a(\|x\|)$$

and

$$\alpha_b = \sup_{t \geq 0} b(t).$$

From (3.2) and the continuity of  $a$  and  $b$  we have  $\alpha_r$  and  $\alpha_b$  are finite.

It yields, by taking into account assumption (A2),

$$\|x(t)\|^c \leq \frac{c_2}{c_1} \|x(0)\|^c + \frac{\alpha_b}{c_1} \int_0^t (\alpha_r + k \|x(s)\|^c) ds.$$

An application of Gronwall's Lemma shows that the solutions exist for all  $t \geq 0$ .

*Global uniform asymptotic stability:*

Since  $\lim_{t \rightarrow \infty} b(t) = 0$ , there exists a time  $t_1 \geq 0$ , such that  $b(t) \leq \frac{c_3}{2k}$  for all  $t \geq t_1$ .

Hence, by (3.3), we obtain for all  $\|x\| \geq r$  and for all  $t \geq t_1$ ,

$$\dot{V} \leq -c_3 \|x\|^c + \frac{c_3}{2k} a(\|x\|) \leq -\frac{c_3}{2} \|x\|^c.$$

So,

$$\dot{V}(t, x) \leq -\frac{c_3}{2}V(t, x).$$

This implies that the sphere  $S = \{x : \|x\| \leq r\}$  is globally attractive, that is,  $\lim_{t \rightarrow \infty} d(x(t), S) = 0$  ( $d$  is the distance<sup>1</sup> between  $x$  and  $S$ ). Thus, boundedness of solutions follows.

We invoke now Lemma 2.3 to show that the solution  $x(t)$  goes to zero. Put  $x_0 = x(t_0)$ . Since the solutions are bounded, given any  $c' > 0$ , there exists  $\beta_1 > 0$ , such that for all  $|x_0| < c'$  we have  $|x(t)| < \beta_1$  for all  $t \geq t_0$ . From now on, we fix an arbitrary constant  $c' > 0$  and  $x_0$  such that  $|x_0| < c'$ .

Define:

$$M_a = \sup_{|x| < \beta_1} a(\|x\|).$$

By continuity of  $a(\cdot)$  on  $\mathbb{R}^+$ ,  $M_a$  is finite. Observe also that  $M_a$  is independent of  $x_0$  for  $|x_0| < c'$ .

Using (3.1) and (3.4), we obtain

$$\dot{V}(t, x) \leq -\frac{c_3}{c_2}V(t, x) + M_a b(t). \tag{3.5}$$

Let  $\epsilon > 0$  be arbitrary. We shall prove that there exists a time  $T > 0$  such that  $\|x(t)\| < \epsilon$ , for all  $t \geq T + t_0$ .

First notice that, since  $\lim_{t \rightarrow \infty} b(t) = 0$ , there exists a time  $T_1$ , such that

$$M_a b(t) \leq \frac{1}{2} \frac{c_3 c_1}{c_2} \epsilon^c, \quad \forall t \geq T_1. \tag{3.6}$$

Integrating inequality (3.5) from  $u \in [T_1, t]$  to  $t \geq T_1$ , on both sides of the inequality, we get

$$V(t, x(t)) - V(u, x(u)) \leq \int_u^t \frac{c_3}{c_2} \left( -V(s, x(s)) + \frac{\epsilon^c c_1}{2} \right) ds \tag{3.7}$$

Using Lemma 2.3, the inequality above implies that,

$$V(t, x(t)) \leq \frac{\epsilon^c c_1}{2} + V(T_1, x(T_1)) \exp\left(-\frac{c_3}{c_2}(t - T_1)\right) \tag{3.8}$$

$$\leq \frac{\epsilon^c c_1}{2} + c_2 \|x(T_1)\|^c \exp\left(-\frac{c_3}{c_2}(t - T_1)\right) \tag{3.9}$$

$$\leq \frac{\epsilon^c c_1}{2} + c_2 \beta_1^c \exp\left(-\frac{c_3}{c_2}(t - T_1)\right). \tag{3.10}$$

with  $\beta_1 = \|x(T_1)\|$ .

This is because  $x(t)$  is bounded by  $\beta_1$  for all  $t \geq t_0 \geq 0$ .

On the other hand, there exists a time  $T_2$  (which is independent of  $x_0$ ), such that

$$c_2 \beta_1^c \exp\left(-\frac{c_3}{c_2}(t - T_1)\right) \leq \frac{1}{2} \epsilon^c c_1, \quad \forall t \geq T_2. \tag{3.11}$$

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<sup>1</sup> $d(x, S) = \inf_{s \in S} \|x - s\|$

Using inequalities (3.8) and (3.11), we obtain

$$V(t, x(t)) \leq \epsilon^c c_1, \quad \forall t \geq T = \max(T_1, T_2).$$

Using the fact that,

$$c_1 \|x(t)\|^c \leq V(t, x(t)), \quad \forall t \geq T + t_0.$$

We have,

$$\|x(t)\| \leq \epsilon, \quad \forall t \geq T + t_0.$$

Therefore, the present proof is complete.

Following the same analysis as above one can prove the following proposition in the case when we replace  $c_i \|x\|^c$ ,  $i = 1, 2, 3$  by some  $\mathcal{K}_\infty$  functions.

**Proposition 3.2.** *Under assumptions (A'1) below and (A2), the origin  $x = 0$  of (2.1) is globally uniformly asymptotically stable equilibrium point,*

where

(A'1). There exists continuously differentiable  $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$  such that

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{3.12a}$$

$$\nabla_t V + \nabla_x^\top V f(t, x) \leq -\alpha_3(\|x\|) \tag{3.12b}$$

$$|\nabla_x^\top V g(t, x)| \leq a(\|x\|)b(t) \tag{3.12c}$$

for some class  $\mathcal{K}_\infty$  functions  $\alpha_i, i = 1, 2, 3$  and  $a, b$  satisfying (3.2).

### 3.2. Global uniform practical stability

In this section, it is worth to notice that the origin is not required to be an equilibrium point for the system (2.2). This may be in many situations meaningful from a practical point of view specially, when stability for uncertain systems is investigated. As pointed out in [3], necessary conditions for system (2.2) to be globally uniformly practically exponentially stable have been derived in [4] as follows.

**Theorem 3.3.** [4] *Consider system (2.2). Let  $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable function, such that*

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2 \tag{3.13a}$$

$$\nabla_t V + \nabla_x^\top V f(t, x) \leq -c_3 V(t, x) + r \tag{3.13b}$$

for all  $t \geq 0$ ,  $x \in \mathbb{R}^n$ , with  $c_1, c_2, c_3$  are positive constants and  $r$  non negative constant. Then the ball  $B_\alpha$  is globally uniformly exponentially stable, with  $\alpha = \frac{r}{c_1 c_3}$ .

The proof proposed in [4] is very clever but here, using lemma 2.3, we can give a shorter proof.

*Proof.* Indeed, the time derivative of  $V$  along the trajectories of system (2.1) is

$$\dot{V}(t, x) = \nabla_t V + \nabla_x^\top V f(t, x). \tag{3.14}$$

Using equation (3.13b) we get

$$\dot{V}(t, x) \leq -c_3 V(t, x) + r. \tag{3.15}$$

Now, integrating both sides the above inequality from  $t \geq 0$  to  $u \in [0, t]$ , we obtain

$$V(t, x(t)) - V(u, x(u)) \leq \int_u^t (-c_3 V(s, x(s)) + r) ds. \tag{3.16}$$

By applying lemma 2.3 with  $\lambda = c_3$  and  $\varepsilon = r$ , it yields

$$V(t, x(t)) \leq \frac{r}{c_3} + V(0, x(0)) \exp(-\lambda t) \tag{3.17}$$

Which implies that,

$$\begin{aligned} \|x(t)\| &\leq \left( \frac{r}{c_1 c_3} + \frac{1}{c_1} V(0, x(0)) \exp(-\lambda t) \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{r}{c_1 c_3}} + \sqrt{\frac{1}{c_1} V(0, x(0))} \exp\left(-\frac{\lambda}{2} t\right), \\ &\leq \sqrt{\frac{r}{c_1 c_3}} + \sqrt{\frac{1}{c_1} c_2} \|x(0)\| \exp\left(-\frac{\lambda}{2} t\right). \end{aligned}$$

This completes the proof.

The previous theorem can be generalized as follows.

**Theorem 3.4.** Consider system (2.2). Let  $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable function, such that

$$c_1 \|x\|^c \leq V(t, x) \leq c_2 \|x\|^c \tag{3.18a}$$

$$\nabla_t V + \nabla_x^T V f(t, x) \leq -c_3 V(t, x) + r \tag{3.18b}$$

for all  $t \geq 0, x \in \mathbb{R}^n$ , with  $c, c_1, c_2, c_3$  are positive constants and  $r$  a non negative constant. Then,

1. if  $c \geq 1$ , the ball  $B_{\alpha_1}$ , where  $\alpha_1 = \left(\frac{r}{c_1 c_3}\right)^{\frac{1}{c}}$ , is globally uniformly exponentially stable.
2. if  $c \leq 1$ , the ball  $B_{\alpha_2}$ , where  $\alpha_2 = 2^{\frac{1}{c}-1} \left(\frac{r}{c_1 c_3}\right)^{\frac{1}{c}}$ , is globally uniformly exponentially stable.

*Proof.* Following a similar reasoning as above, one can prove easily that

$$\|x(t)\| \leq \left( \frac{r}{c_1 c_3} + c_2 c_1^{-1} \|x(0)\|^c \exp(-\lambda t) \right)^{\frac{1}{c}} \tag{3.19}$$

1. If  $c \geq 1$ , by using the fact that  $(a + b)^\varepsilon \leq a^\varepsilon + b^\varepsilon$ , for all  $a, b \geq 0$  and  $\varepsilon \in ]0, 1]$ , one obtains

$$\|x(t)\| \leq \left( \frac{r}{c_1 c_3} \right)^{\frac{1}{c}} + (c_2 c_1^{-1})^{\frac{1}{c}} \|x(0)\| \exp\left(-\frac{\lambda}{c} t\right) \tag{3.20}$$

2. If  $c \leq 1$ . Since  $(a + b)^p \leq 2^{p-1} (a^p + b^p)$ , for all  $a, b \geq 0$  and  $p \geq 1$ , one can get the conclusion by using a similar reasoning as above. This completes the proof.

Another interesting result relying upon the notion of global uniform asymptotic practical stability is the following theorem.



**Theorem 3.5.** *Consider the nonlinear system (2.2). Assume that there exists a continuously differentiable function  $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \quad (3.21a)$$

$$\nabla_t V + \nabla_x^\top V f(t, x) \leq -\alpha_3(V(t, x)) + c \quad (3.21b)$$

for some class  $\mathcal{K}_\infty$  functions  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  and  $c \geq 0$ . Then the ball  $B_r$  is globally uniformly asymptotically practically stable, where  $r = \alpha_1^{-1}(\alpha_3^{-1}(2c))$ .

*Proof.* Let  $x_0 \in \mathbb{R}^n$  and consider the trajectory  $x(t)$  with the initial condition  $x(0) = x_0$  and define  $y(t) = V(t, x(t))$ .

Equation (3.21b) implies that

$$\dot{y}(t) \leq -\alpha_3(y(t)) + c \quad (3.22)$$

Hence, from lemma 2.4 one can deduce the existence of a class  $\mathcal{KL}$  function  $\beta$  such that

$$V(t, x(t)) \leq \alpha_3^{-1}(2c) + \beta\left(V(0, x_0), t\right). \quad (3.23)$$

Put  $\beta_d(r, s) = \beta(\alpha_2(r), s)$  which is a class  $\mathcal{K}_\infty$  function. Thus, using (3.21a), the inequality above implies that

$$\|x(t)\| \leq \alpha_1^{-1}\left(\alpha_3^{-1}(2c) + \beta_d(\|x_0\|, t)\right) \quad (3.24)$$

$$\leq \alpha_1^{-1}\left(\alpha_3^{-1}(2c)\right) + \alpha_1^{-1}\left(2\beta_d(\|x_0\|, t)\right). \quad (3.25)$$

Here, we use the following general fact, a weak form of the triangle inequality which holds for any function  $\gamma$  of class  $\mathcal{K}$  and any  $a, b \geq 0$  (see Lemma 3):

$$\gamma(a + b) \leq \gamma(2a) + \gamma(2b). \quad (3.26)$$

It then holds that the GUPAS is fulfilled with  $r = \alpha_1^{-1}(\alpha_3^{-1}(2c))$  and

$$\beta(s, t) = \alpha_1^{-1}\left(\beta_d(s, t)\right).$$

**Lemma 3.6.** *let  $\gamma : [0, +\infty[ \rightarrow [0, +\infty[$  a function of class  $\mathcal{K}$ . Then, for all  $x, y \geq 0$ , we have:*

$$\gamma(x + y) \leq \gamma(2x) + \gamma(2y).$$

*Proof.* Clearly, if  $x = 0$  or  $y = 0$ , then (3.26) holds. Let  $x > 0$  and  $y > 0$  and define the sets  $I_x = [0, x]$ ,  $I_y = [0, y]$  which are compact for every fixed  $x$  and  $y$  (since they are closed and bounded subsets of  $[0, +\infty[ \subset \mathbb{R}$ ). If  $y \leq x$ , the point  $y \in I_x$ . As a consequence, we have

$$\gamma(x + y) \leq \max_{s \in I_x} \gamma(x + s) := \gamma_1(x).$$

$\gamma_1$  is well defined due to the continuity of  $\gamma$  and the compactness of  $I_x$ . But, if  $s \in I_x$ , we have  $s + x \leq 2x$ . Since  $\gamma$  is a non decreasing function, then  $\gamma_1(x) = \gamma(2x)$ . If  $x \leq y$ , the point  $x \in I_y$ . By using the same argument as above, we conclude that

$$\gamma(x + y) \leq \max_{t \in I_y} \gamma(y + t) := \gamma_2(y) = \gamma(2y).$$

So, for all  $x \geq 0$  and  $y \geq 0$ ,

$$\gamma(x + y) \leq \gamma(2x) + \gamma(2y).$$

**Example 3.7.** We present now an example that implement the previous theorem. Consider the following planar system:

$$\dot{x} = \begin{pmatrix} -x_1 + \frac{\epsilon x_1}{1+x_1^2} \exp(-x_1^2) \\ -x_2 + \frac{\epsilon x_2}{1+x_2^2} \exp(-x_2^2) \end{pmatrix} + \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} a(t)$$

where  $x = (x_1, x_2)^\top \in \mathbb{R}^2$ ,  $a(t)$  is a continuous bounded function and  $\epsilon > 0$ . Choosing the quadratic Lyapunov function  $V(t, x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ . The time derivative of  $V$  along the trajectories of the system is bounded by

$$\dot{V}(t, x) \leq -2V(t, x) + \epsilon.$$

Let  $\epsilon = \frac{1}{2}$ , by application of theorem 3.3, this planar system is globally practically exponentially stable. Moreover, the ball  $B_{\frac{1}{2}}$  is globally uniformly practically exponentially stable. Now, for  $\epsilon$  small enough, the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner.

**Conclusion.** In this paper new sufficient conditions are established to prove the global uniform practical stability of a certain class of time-varying dynamical system. Moreover, a new proof for the result of [4] is presented. The effectiveness of the conditions obtained in this paper is verified in a numerical example.

**Acknowledgement.** The authors wish to thank the reviewer for his valuable and careful comments.

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