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Improvement of a result due to P.T. Mocanu

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Abstract. A result concerning the starlikeness of the image of the Alexander operator is improved in this paper. The techniques of differential subordinations are used.

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1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in in the complex plane. Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disk U and have the properties $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, consisting of functions for which the domain $f(U)$ is starlike with respect to 0 is denoted by S^*. An analytic characterization of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ we deal with is the class of close-to-convex functions denoted by C. A function $f \in \mathcal{A}$ belongs to the class C if and only if there is a starlike function $g \in S^*$, so that $\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U$. We note that C and S^* contain univalent functions. The Alexander integral operator is defined by the equality:

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

The authors of [1] pp. $310 - 311$ proved the following result:

Theorem 1.1. *Let A be the Alexander operator and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|,\ z \in U. \tag{1.1}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U,$$

then $F = A(f) \in S^*$.

Improvements of this result can be found in [3], [4] and [6]. In this paper we put the problem to determine the smallest c_1 such that the following theorems hold.

Theorem 1.2. *Let A be the Alexander operator and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq c_1 \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|, \; z \in U. \tag{1.2}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \; z \in U,$$

then $F = A(f) \in S^*$.

In [5] it has been proved that $A(C) \not\subseteq S^*$, and this result shows that $c_1 > 0$. We are not able to determine the the best value of c_1, but we will give a new improvement for Theorem 1.1 in the present paper. In order to do this, we need some lemmas, which are exposed in the next section.

2. Preliminaries

Let f and g be analytic functions in U. The function f is said to be subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, $z \in U$ and $f(z) = g(w(z))$, $z \in U$. Recall that if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 2.1. [1] (Miller-Mocanu) *Let $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$ be analytic in U with $p(z) \not\equiv a$, $n \geq 1$ and let $q : U \to \mathbb{C}$ be an analytic and univalent function with $q(0) = a$. If p is not subordinate to q, then there are two points $z_0 \in U$, $|z_0| = r_0$ and $\zeta_0 \in \partial U$ and a real number $m \in [n, \infty)$, so that q is defined in ζ_0, $p(U(0, r_0)) \subset q(U)$, and:*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$

and

(iii) $\operatorname{Re} \left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \operatorname{Re} \left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right)$.

We note that $z_0 p'(z_0)$ is the outward normal to the curve $p(\partial U(0, r_0))$ at the point $p(z_0)$, while $\partial U(0, r_0)$ denotes the border of the disc $U(0, r_0)$.

In [6] the following result is proved:

Lemma 2.2. [6] *Let $g \in \mathcal{A}$ be a function, which satisfies the condition*

$$\left| \operatorname{Im} \frac{zg'(z)}{g(z)} \right| \leq 1, \; z \in U. \tag{2.1}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \; z \in U,$$

then $F = A(f) \in S^*$.

3. The main result

Theorem 3.1. *Let $g \in \mathcal{A}$ be a function such that*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \frac{2}{5} \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|, \ z \in U. \tag{3.1}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

Proof. If we denote $p(z) = \frac{zg'(z)}{g(z)}$, then (3.1) becomes

$$\operatorname{Re} p(z) > \frac{2}{5} \left| \operatorname{Im}[zp'(z) + p^2(z)] \right|, \quad z \in U. \tag{3.2}$$

We will prove that

$$p \prec q \ \text{ where } \ q(z) = 1 + \frac{2}{\pi} \log \frac{1+z}{1-z}, \ z \in U.$$

If the subordination $p \prec q$ does not hold, then according to Lemma 2.1, there are two points $z_2 \in U$, $\zeta_2 = e^{i\theta_2}$ and a real number $m_2 \in [1, \infty)$ such that

$$p(z_2) = q(\zeta_2) = 1 + \frac{2}{\pi} \log \frac{1+\zeta_2}{1-\zeta_2} = 1 + \frac{2}{\pi} (\ln|\cot \frac{\theta_2}{2}| \pm i \frac{\pi}{2})$$

and

$$z_2 p'(z_2) = m_2 \zeta_2 q'(\zeta_2) = \frac{2m_2 i}{\pi \sin \theta_2}.$$

We discuss the case $\theta_2 \in (0, \pi)$, the other case $\theta_2 \in [-\pi, 0)$ is similar. If $\theta_2 \in [0, \pi]$ and $x = \cot \frac{\theta_2}{2}$, then

$$p(z_2) = 1 + \frac{2}{\pi} \left(\ln|\cot \frac{\theta_2}{2}| + i \frac{\pi}{2} \right)$$

and we get

$$\left| \operatorname{Im}[z_2 p'(z_2) + p^2(z_2)] \right| - \frac{5}{2} \operatorname{Re} p(z_2)$$

$$= \frac{2m_2}{\pi \sin \theta_2} + 2 \left[1 + \frac{2}{\pi} \ln \left(\cot \frac{\theta_2}{2} \right) \right] - \frac{5}{2} \left[1 + \frac{2}{\pi} \ln \left(\cot \frac{\theta_2}{2} \right) \right]$$

$$\geq \frac{1+x^2}{\pi x} - \frac{1}{2} \big[1 + \frac{2}{\pi} \ln(x) \big] = \frac{1+x^2}{\pi x} - \frac{1}{2} - \frac{1}{\pi} \ln(x) \geq 0, \ x \in (0, \infty).$$

This contradicts (3.2), and consequently the subordination

$$\frac{zg'(z)}{g(z)} = p(z) \prec q(z) = 1 + \frac{2}{\pi} \log \frac{1+z}{1-z}$$

holds. This subordination implies $\left| \operatorname{Im} \frac{zg'(z)}{g(z)} \right| \leq 1, \ z \in U$ and so according to Lemma 2.2 we have $F = A(f) \in S^*$. □

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