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On some integral inequalities for twice differentiable mappings

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Abstract. In this paper, we establish several new inequalities for some twice differantiable mappings. Then, we apply these inequalities to obtain new midpoint, trapezoid and perturbed trapezoid rules. Finally, some applications for special means of real numbers are provided.

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1. Introduction

In 1938 Ostrowski obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is well known in the literature as Ostrowski's integral inequality [14]:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

In 1976, Milovanović and Pečarić proved a generalization of the Ostrowski inequality for n-times differentiable mappings (see for example [13, p.468]). Dragomir and Wang ([10], [11]) extended the result (1.1) and applied the extended result to numerical quadrature rules and to the estimation of error bounds for some special means. Also, Sofo and Dragomir [18] extended the result (1.1) in the L_p norm. Dragomir ([6]-[8]) further extended the (1.1) to incorporate mappings of bounded variation, Lipschitzian and monotonic mappings. For recent results and generalizations conserning Ostrowski's integral inequality see [1]-[13], [18], [19], and the references therein.

In [4], Cerone and Dragomir find the following perturbed trapezoid inequalities:

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then, the inequality holds:*

$$\left| \int_a^b f(t)dt - \frac{b-a}{2}\left[f(b)+f(a)\right] + \frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right] \right| \tag{1.2}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24}\left\|f''\right\|_\infty & \text{if} \quad f'' \in L_\infty[a,b] \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f''\right\|_p & \text{if} \quad f'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \frac{(b-a)^2}{8}\left\|f''\right\|_1 & \text{if} \quad f'' \in L_1[a,b] \end{cases}$$

for all $t \in [a,b]$.

In recent years a number of authors have considered an error analysis for some known and some new quadrature formulas. They used an approach from the inequalities point of view. For example, the midpoint quadrature rule is considered in [4],[15],[17], the trapezoid rule is considered in [4],[16],[20]. In most cases estimations of errors for these quadrature rules are obtained by means of derivatives and integrands.

In this article, we first derive a general integral identity for twice derivatives functions. Then, we apply this identity to obtain our results and using functions whose twice derivatives in absolute value at certain powers are convex, we obtained new inequalities related to the Ostrowski's type inequality. Finally, we gave some applications for special means of real numbers.

2. Main results

In order to prove our main results, we need the following Lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a,b]$, then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}\left[f(x)+f(a+b-x)\right] \\ &+\frac{1}{2}(x-\frac{a+3b}{4})\left[f'(x)-f'(a+b-x)\right] \\ &= \frac{(b-a)^2}{2}\int_0^1 k(t)\, f''(ta+(1-t)b)dt \end{aligned} \tag{2.1}$$

where

$$k(t) := \begin{cases} t^2, & 0 \le t < \frac{b-x}{b-a} \\ \left(t - \frac{1}{2}\right)^2, & \frac{b-x}{b-a} \le t < \frac{x-a}{b-a} \\ (t-1)^2, & \frac{x-a}{b-a} \le t \le 1 \end{cases}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 k(t)\, f''(ta + (1-t)b)dt \\ &= \int_0^{\frac{b-x}{b-a}} t^2 f''(ta + (1-t)b)dt + \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2}\right)^2 f''(ta + (1-t)b)dt \\ &\quad + \int_{\frac{x-a}{b-a}}^{1} (t-1)^2 f''(ta + (1-t)b)dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By inegration by parts, we have the following identity

$$\begin{aligned} I_1 &= \int_0^{\frac{b-x}{b-a}} t^2 f''(ta + (1-t)b)dt \\ &= \frac{t^2}{(a-b)} f'(ta + (1-t)b) \Big|_0^{\frac{b-x}{b-a}} - \frac{2}{a-b} \int_0^{\frac{b-x}{b-a}} t f'(ta + (1-t)b)dt \\ &= \frac{1}{(a-b)} \left(\frac{b-x}{b-a}\right)^2 f'(x) \\ &\quad - \frac{2}{a-b} \left[\frac{t}{(a-b)} f(ta + (1-t)b) \Big|_0^{\frac{b-x}{b-a}} - \frac{1}{a-b} \int_0^{\frac{b-x}{b-a}} f(ta + (1-t)b)dt \right] \\ &= -\frac{(b-x)^2}{(b-a)^3} f'(x) - \frac{2(b-x)}{(b-a)^3} f(x) + \frac{2}{(b-a)^2} \int_0^{\frac{b-x}{b-a}} f(ta + (1-t)b)dt. \end{aligned}$$

Similarly, we observe that

$$I_2 = \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2}\right)^2 f''(ta + (1-t)b)dt$$

$$= \frac{(a+b-2x)^2}{4(b-a)^3} \left[f'(x) - f'(a+b-x)\right] + \frac{(a+b-2x)}{(b-a)^3} \left[f(x) + f(a+b-x)\right]$$

$$+\frac{2}{(b-a)^2}\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} f(ta+(1-t)b)dt$$

and

$$I_3=\int_{\frac{x-a}{b-a}}^{1}(t-1)^2 f''(ta+(1-t)b)dt$$

$$=\frac{(b-x)^2}{(b-a)^3}f'(a+b-x)-\frac{2(b-x)}{(b-a)^3}f(a+b-x)+\frac{2}{(b-a)^2}\int_{\frac{x-a}{b-a}}^{1} f(ta+(1-t)b)dt.$$

Thus, we can write

$$I=I_1+I_2+I_3=\frac{1}{(b-a)^2}\left(x-\frac{a+3b}{4}\right)\left[f'(x)-f'(a+b-x)\right]$$

$$-\frac{1}{(b-a)^2}\left[f(x)+f(a+b-x)\right]+\frac{2}{(b-a)^2}\int_0^1 f(ta+(1-t)b)dt.$$

Using the change of the variable $u=ta+(1-t)b$ for $t\in[0,1]$ and by multiplying the both sides by $(b-a)^2/2$ which gives the required identity (2.1). □

Now, by using the above lemma, we prove our main theorems:

Theorem 2.2. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L_1[a,b]$ where $a,b\in I$, $a<b$. If $|f''|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-\frac{a+3b}{4})\left[f'(x)-f'(a+b-x)\right]\right|$$

$$\leq\frac{1}{(b-a)}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right]\left(\frac{|f''(a)|+|f''(b)|}{6}\right) \tag{2.2}$$

for any $x\in\left[\frac{a+b}{2},b\right]$.

Proof. From Lemma 2.1 and by the definition $k(t)$, we get

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}\left(x-\frac{a+3b}{4}\right)\left[f'(x)-f'(a+b-x)\right]\right|$$

$$\leq\frac{(b-a)^2}{2}\int_0^1|k(t)||f''(ta+(1-t)b)|dt$$

$$=\frac{(b-a)^2}{2}\left\{\int_0^{\frac{b-x}{b-a}}t^2|f''(ta+(1-t)b)|dt+\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left(t-\frac{1}{2}\right)^2|f''(ta+(1-t)b)|dt\right.$$

$$\left.+\int_{\frac{x-a}{b-a}}^{1}(t-1)^2|f''(ta+(1-t)b)|dt\right\}$$

$$=\frac{(b-a)^2}{2}\left\{J_1+J_2+J_3\right\} \tag{2.3}$$

Investigating the three separate integrals, we may evaluate as follows:

By the convexity of $|f''|$, we arrive at

$$
\begin{aligned}
J_1 &\leq \int_0^{\frac{b-x}{b-a}} \left(t^3 |f''(a)| + (t^2 - t^3) |f''(b)|\right) dt \\
&= \frac{(b-x)^4}{4(b-a)^4} |f''(a)| + \left(\frac{(b-x)^3}{3(b-a)^3} - \frac{(b-x)^4}{4(b-a)^4}\right) |f''(b)|,
\end{aligned}
$$

$$
\begin{aligned}
J_2 &\leq \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left[\left(t - \frac{1}{2}\right)^2 t |f''(a)| + \left(t - \frac{1}{2}\right)^2 (1-t) |f''(b)|\right] dt \\
&= \frac{1}{3(b-a)^3} \left(x - \frac{a+b}{2}\right)^3 |f''(a)| + \frac{1}{3(b-a)^3} \left(x - \frac{a+b}{2}\right)^3 |f''(b)|,
\end{aligned}
$$

$$
\begin{aligned}
J_3 &\leq \int_{\frac{x-a}{b-a}}^{1} \left[(t-1)^2 t |f''(a)| + (1-t)^3 |f''(b)|\right] dt \\
&= \left(\frac{(b-x)^3}{3(b-a)^3} - \frac{(b-x)^4}{4(b-a)^4}\right) |f''(a)| + \frac{(b-x)^4}{4(b-a)^4} |f''(b)|.
\end{aligned}
$$

By rewrite J_1, J_2, J_3 in (2.3), we obtain (2.2) which completes the proof. □

Corollary 2.3 (Perturbed Trapezoid inequality). *Under the assumptions Theorem* 2.2 *with* $x = b$, *we have*

$$
\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(b)+f(a)}{2} + \frac{(b-a)}{8}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^2}{48}(|f''(a)| + |f''(b)).
$$

Remark 2.4. We choose $|f''(x)| \leq M$, $\ M > 0$ in Corollary 2.3, then we recapture the first part of the inequality (1.2).

Corollary 2.5 (Trapezoid inequality). *Under the assumptions Theorem* 2.2 *with* $x = b$ *and* $f'(a) = f'(b)$ *in Theorem* 2.2, *we have*

$$
\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{48}(|f''(a)| + |f''(b)|). \tag{2.4}
$$

Corollary 2.6 (Midpoint inequality). *Under the assumptions Theorem* 2.2 *with* $x = \frac{a+b}{2}$ *in Theorem* 2.2, *we have*

$$
\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{48}(|f''(a)| + |f''(b)|). \tag{2.5}
$$

Another similar result may be extended in the following theorem

Theorem 2.7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a,b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(x)+f(a+b-x)]\right. \\
&\left.+\frac{1}{2}\Big(x-\frac{a+3b}{4}\Big)[f'(x)-f'(a+b-x)]\right| \\
&\leq \frac{2^{\frac{1}{p}-1}}{(2p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}}\Bigg[(b-x)^{2p+1} \\
&+\Big(x-\frac{a+b}{2}\Big)^{2p+1}\Bigg]^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.6}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$.

Proof. From Lemma 2.1, by the definition $k(t)$ and using Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(x)+f(a+b-x)]\right. \\
&\left.+\frac{1}{2}\Big(x-\frac{a+3b}{4}\Big)[f'(x)-f'(a+b-x)]\right| \\
&\leq \frac{(b-a)^2}{2}\int_0^1 |k(t)||f''(ta+(1-t)b)|dt \\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 |k(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.7}$$

Since $|f''|^q$ is convex on $[a,b]$, we know that for $t \in [0,1]$

$$|f''(ta+(1-t)b)|^q \leq t\,|f''(a)|^q + (1-t)\,|f''(b)|^q,$$

hence, a simple computation shows that

$$\int_0^1 |f''(ta+(1-t)b)|^q\,dt \leq \frac{|f''(a)|^q+|f''(b)|^q}{2} \tag{2.8}$$

also,

$$\begin{aligned}
\int_0^1 |k(t)|^p\,dt &= \int_0^{\frac{b-x}{b-a}} t^{2p}dt + \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left|t-\frac{1}{2}\right|^{2p} dt + \int_{\frac{x-a}{b-a}}^1 (1-t)^{2p}\,dt \\
&= \frac{2}{(2p+1)(b-a)^{2p+1}}\left[(b-x)^{2p+1}+\Big(x-\frac{a+b}{2}\Big)^{2p+1}\right].
\end{aligned} \tag{2.9}$$

Using (2.8) and (2.9) in (2.7), we obtain (2.6). □

Corollary 2.8 (Perturbed Trapezoid inequality). *Under the assumptions Theorem* 2.7 *with* $x=b$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{f(b)+f(a)}{2}+\frac{(b-a)}{8}[f'(b)-f'(a)]\right|$$
$$\leq\frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 2.9 (Trapezoid inequality). *Under the assumptions Theorem* 2.7 *with* $x=b$ *and* $f'(a)=f'(b)$ *in Theorem* 2.7, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{f(a)+f(b)}{2}\right|\tag{2.10}$$
$$\leq\frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 2.10 (Midpoint inequality). *Under the assumptions Theorem* 2.7 *with* $x=\frac{a+b}{2}$ *in Theorem* 2.7, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{8\,(2p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.\tag{2.11}$$

Theorem 2.11. *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a twice differentiable function on* I° *such that* $f''\in L_1[a,b]$ *where* $a,b\in I$, $a<b$. *If* $|f''|^q$ *is convex on* $[a,b]$ *and* $q\geq 1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}[f(x)+f(a+b-x)]\right.$$
$$\left.+\frac{1}{2}(x-\frac{a+3b}{4})[f'(x)-f'(a+b-x)]\right|\tag{2.12}$$
$$\leq\frac{1}{3(b-a)}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right]\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}$$

for any $x\in\left[\frac{a+b}{2},b\right]$.

Proof. From Lemma 2.1, by the definition $k(t)$ and using power mean inequality, it follows that

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}[f(x)+f(a+b-x)]\right.$$
$$\left.+\frac{1}{2}(x-\frac{a+3b}{4})[f'(x)-f'(a+b-x)]\right|\tag{2.13}$$
$$\leq\frac{(b-a)^2}{2}\int_0^1|k(t)||f''(ta+(1-t)b)|dt$$
$$\leq\frac{(b-a)^2}{2}(\int_0^1|k(t)|dt)^{1-\frac{1}{q}}(\int_0^1|k(t)||f''(ta+(1-t)b)|^q dt)^{\frac{1}{q}}.$$

Since $|f''|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$|f''(ta+(1-t)b)|^q \le t\,|f''(a)|^q+(1-t)\,|f''(b)|^q\,,$$

hence, by simple computation

$$\begin{aligned}\int_0^1 |k(t)|\,dt &= \int_0^{\frac{b-x}{b-a}} t^2dt+\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left|t-\frac{1}{2}\right|^2dt+\int_{\frac{x-a}{b-a}}^1(1-t)^2\,dt \\ &= \frac{2}{3\,(b-a)^3}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right],\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}&\int_0^1 |k(t)|\,|f''(ta+(1-t)b)|^q\,dt \\ \le\ & \int_0^1 |k(t)|\left(t\,|f''(a)|^q+(1-t)\,|f''(b)|^q\right)dt \\ =\ & \int_0^{\frac{b-x}{b-a}}\left(t^3\,|f''(a)|^q+(t^2-t^3)\,|f''(b)|^q\right)dt \\ & +\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left[\left(t-\frac{1}{2}\right)^2 t\,|f''(a)|^q+\left(t-\frac{1}{2}\right)^2(1-t)\,|f''(b)|^q\right]dt \\ & +\int_{\frac{x-a}{b-a}}^1\left[(t-1)^2\,t\,|f''(a)|^q+(1-t)^3\,|f''(b)|^q\right]dt \\ =\ & \frac{(b-x)^4}{4(b-a)^4}\,|f''(a)|^q+\left(\frac{(b-x)^3}{3(b-a)^3}-\frac{(b-x)^4}{4(b-a)^4}\right)|f''(b)|^q \\ & +\frac{1}{3(b-a)^3}\left(x-\frac{a+b}{2}\right)^3|f''(a)|^q+\frac{1}{3(b-a)^3}\left(x-\frac{a+b}{2}\right)^3|f''(b)|^q \\ & +\left(\frac{(b-x)^3}{3(b-a)^3}-\frac{(b-x)^4}{4(b-a)^4}\right)|f''(a)|^q+\frac{(b-x)^4}{4(b-a)^4}\,|f''(b)|^q \\ =\ & \frac{1}{3\,(b-a)^3}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right]\left(|f''(a)|^q+|f''(b)|^q\right).\end{aligned}\tag{2.15}$$

Using (2.14) and (2.15) in (2.13), we obtain (2.12). □

Corollary 2.12. *Under the assumptions Theorem* 2.11 *with* $x = b$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(b)+f(a)}{2} + \frac{(b-a)}{8}[f'(b)-f'(a)]\right|$$
$$\leq \frac{(b-a)^2}{24}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 2.13. *Under the assumptions Theorem* 2.11 *with* $x = b$ *and* $f'(a) = f'(b)$ *in Theorem* 2.11, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{24}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.16}$$

Corollary 2.14. *Under the assumptions Theorem* 2.11 *with* $x = \frac{a+b}{2}$ *in Theorem* 2.11, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.17}$$

Remark 2.15. When considering that $24 > 8(2p+1)^{\frac{1}{p}}$, $p > 1$, the bounded of Corrolaries 2.12-2.14 are better than Corrolaries 2.8-2.10 respectively.

Open Problem. Under what conditions that the result of Theorem 2.7 and Theorem 2.11 is comparable.

3. Applications for special means

Recall the following means:

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \ a, b \geq 0;$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab}, \ a, b \geq 0;$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \ a, b > 0;$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & if \quad a = b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \neq b \end{cases}, \qquad a, b > 0;$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & if \quad a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & if \quad a \neq b \end{cases}, \qquad a, b > 0;$$

(f) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \qquad p \in \mathbb{R} \diagdown \{-1, 0\}\,;\ a, b > 0.$$

It is also known that L_p is monotonically nondecreasing in $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 3.1. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality:*

$$\left|L_p^p(a,b) - A\left(a^p, b^p\right)\right| \leq p\,(p-1)\,\frac{(b-a)^2}{24} A\left(a^{p-2}, b^{p-2}\right).$$

Proof. The assertion follows from (2.4) applied for $f(x) = x^p$, $x \in [a,b]$ .We omitted the details. □

Proposition 3.2. *Let $0 < a < b$. Then we have the inequality:*

$$\left|L^{-1}(a,b) - A^{-1}(a,b)\right| \leq \frac{(b-a)^2}{12} A\left(a^{-3}, b^{-3}\right).$$

Proof. The assertion follows from (2.5) applied for $f(x) = \frac{1}{x}$, $x \in [a,b]$. We omitted the details. □

Proposition 3.3. *Let $q > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left|\ln I(a,b) - \ln G(a,b)\right| \leq \frac{(b-a)^2}{8\,(2p+1)^{\frac{1}{p}}} \left[A\left(a^{-2q}, b^{-2q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.10) applied for $f(x) = -\ln x$, $x \in [a,b]$. □

Proposition 3.4. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality:*

$$\left|L_p^p(a,b) - A^p(a,b)\right| \leq p\,(p-1)\,\frac{(b-a)^2}{8\,(2p+1)^{1/p}} \left[A\left(a^{q(p-2)}, b^{q(p-2)}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.11) applied for $f(x) = x^p$, $x \in [a,b]$. □

Proposition 3.5. *Let $p > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left|L^{-1}(a,b) - H^{-1}(a,b)\right| \leq \frac{(b-a)^2}{12} \left[A\left(a^{-3q}, b^{-3q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.16) applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$. $\square$

Proposition 3.6. *Let $q > 1$, and $0 < a < b$. Then we have the inequality:*

$$|\ln I(a,b) - \ln A(a,b)| \leq \frac{(b-a)^2}{24} \left[A\left(a^{-2q}, b^{-2q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.17) applied for $f(x) = -\ln x$, $x \in [a, b]$. $\square$

4. Applications for composite quadrature formula

Let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and $\xi = (\xi_0, ..., \xi_{n-1})$ a sequence of intermediate points, $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$. Then the following result holds:

Theorem 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|$ is convex on $[a, b]$ then we have*

$$\int_a^b f(u)du = A(f, f', d, \xi) + R(f, f', d, \xi)$$

where

$$\begin{aligned} A(f, f', d, \xi) \quad : \quad &= \sum_{i=0}^{n-1} \frac{h_i}{2} \left[f(\xi_i) + f(x_i + x_{i+1} - \xi_i)\right] \\ &- \sum_{i=0}^{n-1} \frac{h_i}{2} \left(\xi_i - \frac{x_i + 3x_{i+1}}{4}\right) \left[f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)\right]. \end{aligned}$$

The remainder $R(f, f', d, \xi)$ satisfies the estimation:

$$|R(f, f', d, \xi)| \leq \sum_{i=0}^{n-1} \left[(x_{i+1} - \xi_i)^3 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right] \left(\frac{|f''(x_i)| + |f''(x_{i+1})|}{6}\right) \tag{4.1}$$

for any choice ξ of the intermediate points.

Proof. Apply Theorem 2.2 on the interval $[x_i, x_{i+1}]$, $i = \overline{0, n-1}$ to get

$$\begin{aligned} &\left| \frac{h_i}{2} \left[f(\xi_i) + f(x_i + x_{i+1} - \xi_i)\right] - \frac{h_i}{2} \left(\xi_i - \frac{x_i + 3x_{i+1}}{4}\right) \right. \\ &\qquad \left. \times \left[f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)\right] - \int_a^b f(u)du \right| \\ &\qquad \leq \left[(x_{i+1} - \xi_i)^3 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right] \left(\frac{|f''(x_i)| + |f''(x_{i+1})|}{6}\right). \end{aligned}$$

$\square$

Summing the above inequalities over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (4.1).

Corollary 4.2. *The following perturbed trapezoid rule holds:*

$$\int_a^b f(u)du = T(f,f',d) + R_T(f,f',d)$$

where

$$T(f,f',d) := \sum_{i=0}^{n-1} \frac{h_i}{2}\left[f(x_i)+f(x_{i+1})\right] - \sum_{i=0}^{n-1} \frac{(h_i)^2}{8}\left[f'(x_{i+1})-f'(x_i)\right]$$

and the remainder term $R_T(f,f',d)$ satisfies the estimation,

$$R_T(f,f',d) \le \sum_{i=0}^{n-1} \frac{(h_i)^3}{48}(|f''(x_i)|+|f''(x_{i+1})|).$$

Corollary 4.3. *The following midpoint rule holds:*

$$\int_a^b f(u)du = M(f,d) + R_M(f,d)$$

where

$$M(f,d) := \sum_{i=0}^{n-1} h_i\left[f(\frac{x_i+x_{i+1}}{2})\right]$$

and the remainder term $R_M(f,d)$ satisfies the estimation,

$$R_M(f,d) \le \sum_{i=0}^{n-1} \frac{(h_i)^3}{48}(|f''(x_i)|+|f''(x_{i+1})|).$$

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