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Some remarks on restriction maps between cohomology of fusion systems

Constantin-Cosmin Todea

Abstract. We define a restriction map between two cohomology algebras of some saturated fusion systems which are chosen in a particular situation. We find conditions for this map to induce an injective map between the varieties which can be associated to these finitely generated graded commutative cohomology algebras. Some minimal examples for which we can apply our results are also given.

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1. Preliminaries

Saturated fusion systems on finite p-groups are intensively studied in the last years by mathematicians from different areas such as: modular representation theory, algebraic topology and finite groups. A saturated fusion system $\mathcal{F}$ on a finite p-group P is a category whose objects are the subgroups of P and whose morphisms satisfy certain axioms mimicking the behavior of a finite group G having P as a Sylow subgroup. The axioms of saturated fusion systems were invented by Puig in early 1990's. See [1] for a detailed exposition of results and definitions involving fusion systems.

The cohomology algebra of a p-local finite group with coefficients in $\mathbb{F}_p$ is introduced in [3, §5] and is equal with cohomology algebra of a saturated fusion system. Let k be an algebraically closed field of characteristic p. We denote by $\mathrm{H}^*(G,k)$ the cohomology algebra of the group G with trivial coefficients. As in [6] we will use the language of homotopy classes of chain maps (see [6, 2.8]). We denote by $\mathrm{H}^*(\mathcal{F})$ the algebra of stable elements of $\mathcal{F}$, i.e. the cohomology algebra of the saturated fusion system $\mathcal{F}$, which is the subalgebra of $\mathrm{H}^*(P,k)$ consisting of elements $[\zeta] \in \mathrm{H}^*(P,k)$ such that

$$\mathrm{res}^P_Q([\zeta]) = \mathrm{res}_\varphi([\zeta]),$$

for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and any subgroup Q of P. This is the main object of study in this paper. Moreover Broto, Levi and Oliver showed that any saturated fusion system $\mathcal{F}$ has a non-unique $P-P$-biset X with certain properties formulated by Linckelmann and Webb (see [3, Proposition 5.5]). Such a $P-P$-biset X is called a *characteristic biset*. Using this biset, S. Park noticed in [8] a result which says that a saturated fusion system can be realized by a finite group. This finite group is $G = \mathrm{Aut}(X_P)$, that is the group of bijections of the characteristic biset X, preserving the right P-action. So, by [8, Theorem 3], we identify $\mathcal{F}$ with $\mathcal{F}_P(G)$ which is the fusion system on P such that for every $Q, R \leq P$ we have

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \{\varphi : Q \to R \mid \exists x \in G \ s.t. \ \varphi(u) = xux^{-1}, \forall u \in Q\}.$$

Using this identification we will define a restriction map from the cohomology algebra of the group G with coefficients in the field k to the cohomology algebra of the fusion system, $\mathrm{H}^*(\mathcal{F})$. We denote this map by $\rho_{\mathcal{F},G}$, and we have the following proposition.

Proposition 1.1. *Let $\mathcal{F}$ be a saturated fusion system on P and let X be a characteristic $P-P$-biset. Let $G = \mathrm{Aut}(X_P)$ and then we identify $\mathcal{F}$ with $\mathcal{F}_P(G)$. We have $\mathrm{res}^G_P(\mathrm{H}^*(G, k)) \subseteq \mathrm{H}^*(\mathcal{F})$, hence there is a homomorphism of algebras*

$$\rho_{\mathcal{F},G} : \mathrm{H}^*(G, k) \to \mathrm{H}^*(\mathcal{F}),$$

given by $\rho_{\mathcal{F},G}([\zeta]) = \mathrm{res}^G_P([\zeta])$, for any $[\zeta] \in \mathrm{H}^*(G, k)$.

Next we will define the main restriction map of this article, between the cohomology algebras of two saturated fusion systems. This is done by considering the following situation:

Situation $(*)$**.** *Let Q be a finite p-subgroup of a finite p-group P. Let G be a finite group which realizes a saturated fusion system $\mathcal{G}$ on P (i.e. $\mathcal{G} = \mathcal{F}_P(G)$) and $\mathcal{F}$ a fusion subsystem (i.e subcategory and fusion system) of $\mathcal{G}$ on Q. We assume that there is H which realizes $\mathcal{F}$ and $Q \leq H \leq G$.*

The next example assure us that there are cases of saturated fusion systems in Situation $(*)$.

Example 1.2. Let H be a finite subgroup of a finite group G with P a Sylow p-subgroup of G such that $P \cap H \neq \{1\}$. Then $\mathcal{F} = \mathcal{F}_{P\cap H}(H)$ and $\mathcal{G} = \mathcal{F}_P(G)$ are in Situation $(*)$.

It is easy to verify that in Situation $(*)$ the restriction map res^P_Q induces a well-defined homomorphism of algebras

$$\mathrm{res}_{\mathcal{G},\mathcal{F}} : \mathrm{H}^*(\mathcal{G}) \to \mathrm{H}^*(\mathcal{F}),$$

given by $\mathrm{res}_{\mathcal{G},\mathcal{F}}([\zeta]) = \mathrm{res}^P_Q([\zeta])$ for any $[\zeta] \in \mathrm{H}^*(\mathcal{G})$.

Now we set some notations, which are known to appear in the Quillen stratification of V_G ([4, Definition 8.4.4, Theorem 8.5.2]) of group cohomology ring. Let E be a p-subgroup of G. The restriction map $\mathrm{res}^G_E : \mathrm{H}^*(G, k) \to \mathrm{H}^*(E, k)$ induces a map on varieties, which we denote

$$r^*_{G,E} : V_E \to V_G.$$

As usual we define the subvariety of V_E

$$V_E^+ = V_E \setminus \bigcup_{F<E} (\mathrm{res}_F^E)^*(V_F),$$

and denote the subvarieties of V_E

$$V_{G,E} = r_{G,E}^*(V_E),\ V_{G,E}^+ = r_{G,E}^*(V_E^+).$$

Finally we set $W_G(E) = N_G(E)/C_G(E)$, the Weyl group. Similarly to the group cohomology ring case, since $\mathrm{H}^*(\mathcal{F})$ is a graded commutative finitely generated k-algebra we can associate the spectrum of maximal ideals, i.e. the variety denoted $V_{\mathcal{F}}$. Varieties for cohomology algebras of particular cases of saturated fusion systems were studied in [7], for fusion systems associated to block algebras of finite groups. See also [2, Chapter 5] for more results regarding varieties.

Theorem 1.3. *We assume that we are in Situation* $(*)$*.*

(i) *The following diagram is commutative*

$$\begin{array}{ccc} \mathrm{H}^*(G,k) & \xrightarrow{\rho_{\mathcal{G},G}} & \mathrm{H}^*(\mathcal{G}) \\ \downarrow{\scriptstyle \mathrm{res}_H^G} & & \downarrow{\scriptstyle \mathrm{res}_{\mathcal{G},\mathcal{F}}} \\ \mathrm{H}^*(H,k) & \xrightarrow{\rho_{\mathcal{F},H}} & \mathrm{H}^*(\mathcal{F}) \end{array}$$

(ii) *If* $\mathrm{Ker}(\mathrm{res}_{\mathcal{G},\mathcal{F}})$ *has a nilpotent ideal then* $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ *induces a finite surjective map*

$$\mathrm{res}_{\mathcal{G},\mathcal{F}}^* : V_{\mathcal{F}} \to V_{\mathcal{G}}.$$

In Situation $(*)$ if $Q = P$ then the restriction $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ becomes the inclusion map, hence $\mathrm{Ker}(\mathrm{res}_{\mathcal{F},\mathcal{G}})$ is a nilpotent ideal. Therefore exist cases for which Theorem 1.3, (ii) is true. The next definitions allow us to find conditions for which $\mathrm{res}_{\mathcal{G},\mathcal{F}}^*$ is injective.

Definition 1.4. *Let* $\mathcal{G}, \mathcal{F}$ *be two saturated fusion systems in Situation* $(*)$*. We say that the pair* $(\mathcal{F}, H)$ *is* **weakly elementary embedded** *in* $(\mathcal{G}, G)$ *if:*

(1) *Whenever* E *is an elementary abelian* p*-subgroup of* H *then* $W_G(E) \cong W_H(E)$*;*
(2) *If two elementary abelian* p*-subgroups of* H *are* G*-conjugate then they are also* H*-conjugate.*

The main result of this article is the following theorem.

Theorem 1.5. *In Situation* $(*)$ *we assume that* $\rho_{\mathcal{F},H}^*$ *is injective. If* $(\mathcal{F}, H)$ *is weakly elementary embedded in* $(\mathcal{G}, G)$ *then* $\mathrm{res}_{\mathcal{G},\mathcal{F}}^*$ *is injective.*

Using Theorem 1.3, (ii) and Theorem 1.5 it is easy to check the following corollary. The proof is left for the reader.

Corollary 1.6. *We assume that we are under the hypothesis of Theorem* 1.5 *such that* $Q = P$*. Then* $\mathrm{res}_{\mathcal{G},\mathcal{F}}^*$ *is a bijective map.*

We notice from Example 1.2 that there are some minimal examples for which the above theorem and corollary can be applied.

2. Proofs of the results

Proof of Proposition 1.1. Let Q be a subgroup of P, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ and let $[\zeta] \in \mathrm{H}^*(G,k)$. We have to prove that

$$\mathrm{res}^P_Q(\mathrm{res}^G_P([\zeta])) = \mathrm{res}_\varphi(\mathrm{res}^G_P([\zeta])).$$

We denote by $\overline{\varphi} = i_1 \circ \varphi$, where $i_1 : P \to G$ is the inclusion. Then we will prove that

$$\mathrm{res}^G_Q([\zeta]) = \mathrm{res}_{\overline{\varphi}}([\zeta]).$$

We consider S a Sylow p-subgroup of G such that $P \leq S$. Then $\mathcal{F}_P(G)$ is a full subcategory of $\mathcal{F}_S(G)$, hence $\varphi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q,P)$. If we take $\varphi' = i_2 \circ \varphi$, where $i_2 : P \to S$ is the inclusion, then $\varphi' \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q,S)$. By Cartan-Eilenberg stable elements theorem ([5, XII, Theorem 10.1]) we have that

$$\mathrm{res}^S_Q(\mathrm{res}^G_S([\zeta])) = \mathrm{res}_{\varphi'}(\mathrm{res}^G_S([\zeta])).$$

Since $\overline{\varphi} = i_3 \circ \varphi'$, where $i_3 : S \to G$ is the inclusion, we get the above, desired condition.

Proof of Theorem 1.3. (i) is easy to check since we have compositions of restrictions. For (ii) we have that $\mathrm{H}^*(\mathcal{F})$ is a $\rho_{\mathcal{F},H}(\mathrm{H}^*(H,k))$-submodule of $\mathrm{H}^*(Q,k)$. Since $\mathrm{H}^*(Q,k)$ is noetherian as $\mathrm{res}^H_Q(\mathrm{H}^*(H,k))$-module it follows that $\mathrm{H}^*(\mathcal{F})$ is a finitely generated $\rho_{\mathcal{F},H}(\mathrm{H}^*(H,k))$-module. Now $\mathrm{H}^*(H,k)$ is a finitely generated $\mathrm{res}^G_H(\mathrm{H}^*(G,k))$-module. Then we obtain that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as $(\rho_{\mathcal{F},H} \circ \mathrm{res}^G_H)(\mathrm{H}^*(G,k))$-module, hence by (i) we get that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as $(\mathrm{res}_{\mathcal{G},\mathcal{F}} \circ \rho_{\mathcal{G},G})(\mathrm{H}^*(G,k))$-module. Since $(\mathrm{res}_{\mathcal{G},\mathcal{F}} \circ \rho_{\mathcal{G},G})(\mathrm{H}^*(G,k))$ is a subalgebra of $\mathrm{res}_{\mathcal{G},\mathcal{F}}(\mathrm{H}^*(\mathcal{G}))$ we obtain that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as $\mathrm{res}_{\mathcal{G},\mathcal{F}}(\mathrm{H}^*(\mathcal{G}))$-module, thus $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ is a finite map. Now $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ is also a dominant map (see [2, Section 5.4]), because $\mathrm{Ker}(\mathrm{res}_{\mathcal{G},\mathcal{F}})$ is a nilpotent ideal. We conclude that it is surjective, see [2, Theorem 5.4.7].

Proof of Theorem 1.5. Let $m_1, m_2 \in V_{\mathcal{F}}$ such that $\mathrm{res}^*_{\mathcal{G},\mathcal{F}}(m_1) = \mathrm{res}^*_{\mathcal{G},\mathcal{F}}(m_2)$. By Theorem 1.3, (i) we have that

$$\rho^*_{\mathcal{G},G} \circ \mathrm{res}^*_{\mathcal{G},\mathcal{F}} = (\mathrm{res}^G_H)^* \circ \rho^*_{\mathcal{F},H};$$

From [4, Theorem 8.5.2] (Quillen stratification) applied to V_H there is $E_1 \leq H$ an elementary abelian p-subgroup and $\gamma_1 \in V^+_{E_1}$ such that $\rho^*_{\mathcal{F},H}(m_1) = r^*_{H,E_1}(\gamma_1)$. Similarly there is $E_2 \leq H$ an elementary abelian p-subgroup and $\gamma_2 \in V^+_{E_2}$ such that $\rho^*_{\mathcal{F},H}(m_2) = r^*_{H,E_2}(\gamma_2)$, hence

$$((\mathrm{res}^G_H)^* \circ \rho^*_{\mathcal{F},H})(m_1) = ((\mathrm{res}^G_H)^* \circ r^*_{H,E_1})(\gamma_1),$$
$$((\mathrm{res}^G_H)^* \circ \rho^*_{\mathcal{F},H})(m_2) = ((\mathrm{res}^G_H)^* \circ r^*_{H,E_2})(\gamma_2).$$

From the above relations it follows that

$$((\mathrm{res}^G_H)^* \circ r^*_{H,E_1})(\gamma_1) = ((\mathrm{res}^G_H)^* \circ r^*_{H,E_2})(\gamma_2),$$

that is

$$r^*_{G,E_1}(\gamma_1) = r^*_{G,E_2}(\gamma_2) \in V^+_{G,E_1} \cap V^+_{G,E_2},$$

thus E_1, E_2 are G-conjugate, and by Definition 1.4, (2) we get that they are H-conjugate. From this we can choose now $E_1 = E_2 = E$ and $r^*_{G,E}(\gamma_1) = r^*_{G,E}(\gamma_2) \in$

$V_{G,E}^{+}$. By the Quillen stratification for $\mathrm{H}^*(G,k)$ we have $V_{G,E}^{+} \cong V_E^{+}/W_G(E)$ and this inseparable isogeny is given by $r_{G,E}$. We obtain that γ_1, γ_2 are in the same orbit of the action of $W_G(E)$ on V_E^{+}. By Definition 1.4, (1) it follows that γ_1, γ_2 are in the same orbit of the action of $W_H(E)$ on V_E^{+}, then $r_{H,E}^*(\gamma_1) = r_{H,E}^*(\gamma_2)$. We conclude that $\rho_{\mathcal{F},H}^*(m_1) = \rho_{\mathcal{F},H}^*(m_2)$, hence $m_1 = m_2$ since $\rho_{\mathcal{F},H}^*$ is injective.

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Constantin-Cosmin Todea
Department of Mathematics
Technical University of Cluj-Napoca
Str. G. Baritiu 25
400027 Cluj-Napoca, Romania
e-mail: `Constantin.Todea@math.utcluj.ro`